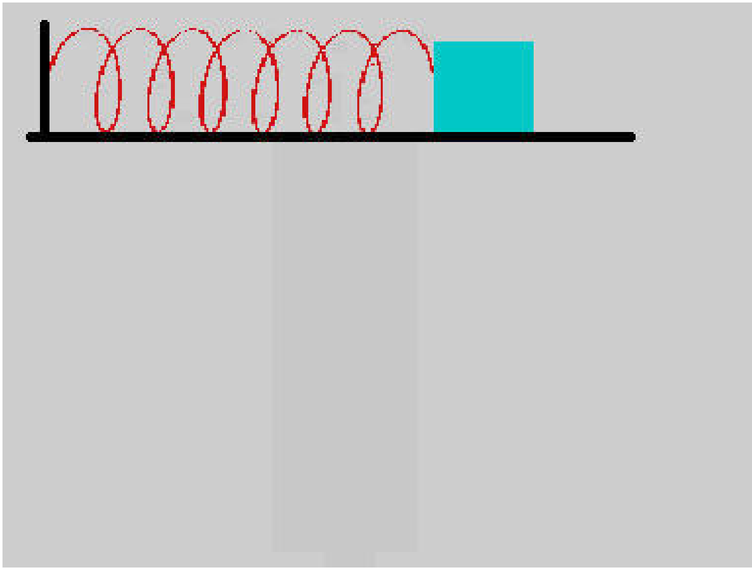
Differential Equations

Dr. E. Jacobs

Today's Topic : The Imaginary Root Case

Review:

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$



$$\frac{d^2y}{dt^2} + 4y = 0$$

Substitute $y = e^{rt}$ into $y'' + 4y = 0$

$$r = \pm 2i$$

The general solution will be a linear combination of e^{2it} and e^{-2it}

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\begin{aligned} y &= c_1 e^{2it} + c_2 e^{-2it} \\ &= c_1(\cos 2t + i\sin 2t) + c_2(\cos 2t - i\sin 2t) \\ &= (c_1 + c_2)\cos 2t + (ic_1 - ic_2)\sin 2t \end{aligned}$$

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$$\frac{d^2y}{dt^2} + 4y = 0$$

$$y = a\cos 2t + b\sin 2t$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

$$y = a \cos \omega t + b \sin \omega t$$

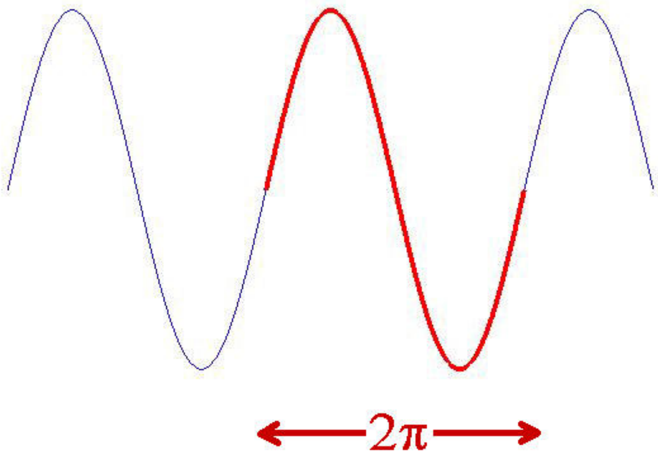
$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

$$y = a\cos\omega t + b\sin\omega t$$

If t is time (in seconds) and ωt is in radians then:

$$\omega = \frac{\text{radians}}{\text{sec}}$$

One cycle of sine or cosine is 2π radians



$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

$$y = a\cos\omega t + b\sin\omega t$$

If t is time (in sec) and ωt is in radians then:

$$\nu = \frac{\omega\,\text{radians}}{\text{sec}} \cdot \frac{1\ \text{cycle}}{2\pi\,\text{radians}} = \frac{\omega}{2\pi}\ \frac{\text{cycles}}{\text{sec}}$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

$$y = a\cos\omega t + b\sin\omega t$$

$$\nu = \frac{\omega\,\text{radians}}{\text{sec}} \cdot \frac{1\ \text{cycle}}{2\pi\,\text{radians}} = \frac{\omega}{2\pi}\ \frac{\text{cycles}}{\text{sec}}$$

$$\tau = \frac{1}{\nu} = \frac{2\pi}{\omega}\ \frac{\text{sec}}{\text{cycle}}$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

If $\beta = 0$ then:

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

This is now in the form $\frac{d^2y}{dt^2} + \omega^2 y = 0$

$$\omega = \sqrt{\frac{k}{m}}$$

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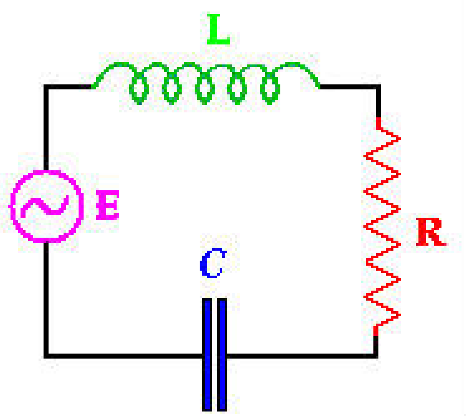
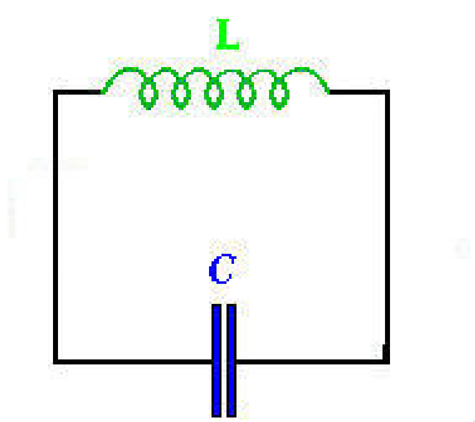
$$\begin{aligned} y &= a\cos\omega t + b\sin\omega t \\ &= a\cos\left(\sqrt{\frac{k}{m}}\,t\right) + b\sin\left(\sqrt{\frac{k}{m}}\,t\right) \end{aligned}$$

$$\nu = \frac{\omega}{2\pi} = \frac{\sqrt{\frac{k}{m}}}{2\pi} \qquad \tau = \frac{1}{\nu} = 2\pi\sqrt{\frac{m}{k}}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)$$



$$L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0$$



$$L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

This is now in the form $y'' + \omega^2 y = 0$

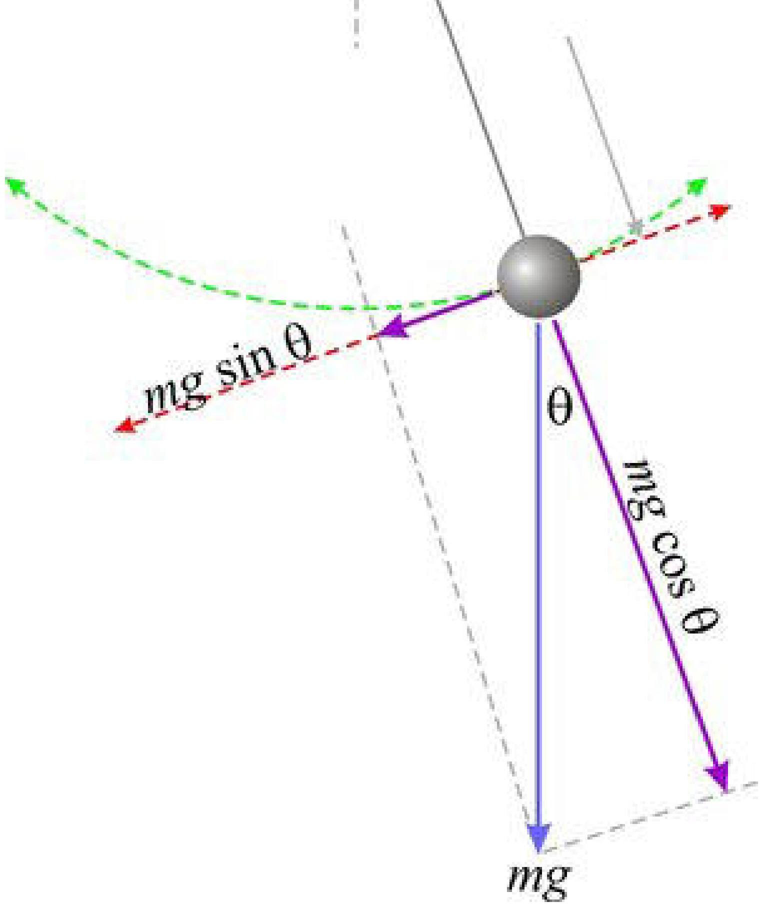
$$\omega = \frac{1}{\sqrt{LC}}$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

$$\begin{aligned} Q &= a\cos\omega t + b\sin\omega t \\ &= a\cos\left(\frac{t}{\sqrt{LC}}\right) + b\sin\left(\frac{t}{\sqrt{LC}}\right) \end{aligned}$$

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$

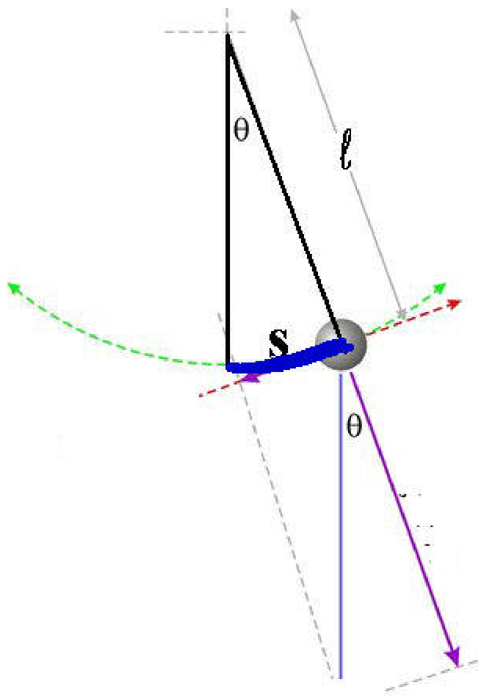
Pendulum



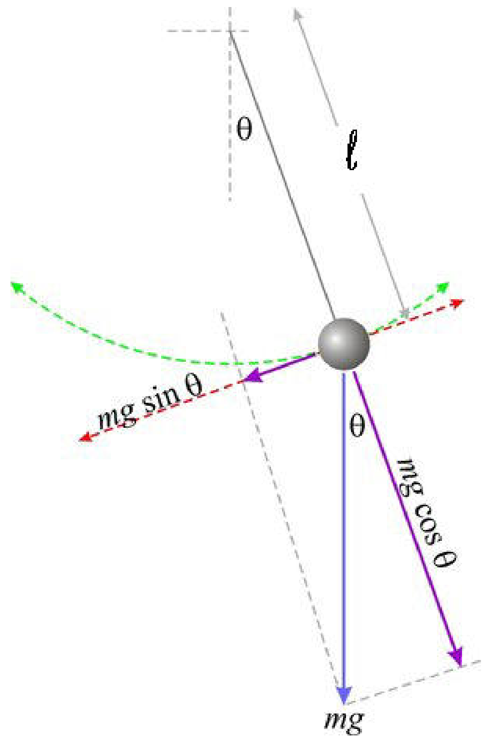
$$\theta = \frac{s}{\ell}$$

$$s = \ell\theta$$

$$\frac{d^2 s}{dt^2} = \ell\frac{d^2\theta}{dt^2}$$



$$F = -mg\sin\theta$$



$$ma = -mg\sin\theta$$

$$m\frac{d^2 s}{dt^2} = -mg\sin\theta$$

$$m\ell\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

This means that for very small oscillations, $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

This is in the form $y'' + \omega^2 y = 0$

$$\omega = \sqrt{\frac{g}{\ell}}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

$$\begin{aligned}\theta &= c_1 \cos \omega t + c_2 \sin \omega t \\ &= c_1 \cos\left(\sqrt{\frac{g}{\ell}}\, t\right) + c_2 \sin\left(\sqrt{\frac{g}{\ell}}\, t\right)\end{aligned}$$

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$$