The Integral Resulting in a Logarithm of a Complex Number
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In first year calculus you learned about the formula:

$$\int_a^b \frac{1}{x}\,dx = \ln|b| - \ln|a|$$

It is natural to want to apply this formula to contour integrals. If z_1 and z_2 are complex numbers and C is a path connecting z_1 to z_2, we would expect that:

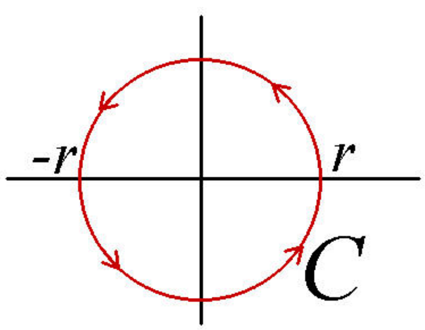
$$\int_C \frac{1}{z}\,dz = \log z_2 - \log z_1$$

However, you are about to see that there is an interesting complication when we take the formula for the integral resulting in logarithm and try to extend it to the complex case.

First of all, recall that if $f(z)$ is an analytic function on a domain $\mathcal{D}$ and C is a closed loop in $\mathcal{D}$ then :

$$\oint f(z)\,dz = 0$$

This is referred to as the *Cauchy-Integral Theorem*. So, for example, if C is a circle of radius r around the origin we may conclude immediately that $\oint_C z^2\,dz = 0$ because z^2 is an analytic function for all values of z.



If $f(z)$ is not analytic then $\oint_C f(z)\,dz$ is *not* necessarily zero. Let's do an explicit calculation for $f(z) = \frac{1}{z}$. If we are integrating around a circle C centered at the origin, then any point z on this path may be written as:

$$z = re^{i\theta}$$

Let's use this to integrate $\oint_C \frac{1}{z}\,dz$. If $z = re^{i\theta}$ then

$$dz = ire^{i\theta}\,d\theta$$

Now, substitute this into the integral $\oint_C \frac{1}{z}\,dz$. If we want to complete one revolution around the circle, let θ vary from $-\pi$ to π.

$$\oint_C \frac{1}{z}\,dz = \int_{-\pi}^{\pi} \frac{1}{re^{i\theta}}\,ire^{i\theta}\,d\theta = \int_{-\pi}^{\pi} i\,d\theta = \Big[i\theta\Big]_{-\pi}^{\pi} = 2\pi i$$

So, the closed loop integral of $\frac{1}{z}$ is not zero. However, this seems to contradict the formula $\int_C \frac{1}{z}\,dz = \log z_2 - \log z_1$. After all, if C is a closed loop then the final point z_2 is the same as the initial point z_1 and any closed loop integral of the form $\oint_C F'(z)\,dz = F(z_2) - F(z_1)$ should be 0, shouldn't it? How can we possibly make sense of a nonzero answer for $\oint_C \frac{1}{z}\,dz$?

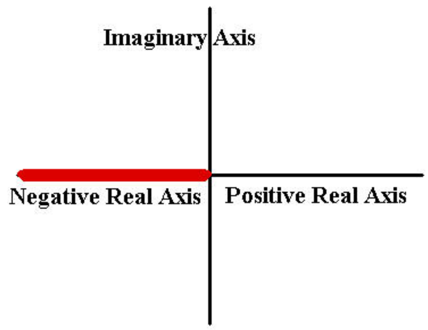
The Logarithm of a Complex Variable

The log function is supposed to be the inverse of the exponential function. Therefore:

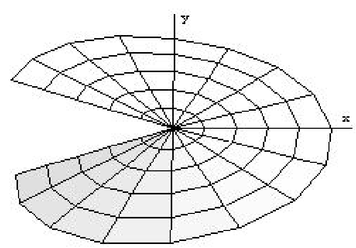
$$\log e^{i\theta} = i\theta$$

For example, $\log e^{i\cdot 0} = i \cdot 0 = 0$ and $\log e^{i\cdot 2\pi} = i \cdot 2\pi$. The problem is that for an operation f to be considered a function then $f(z)$ must have a unique value for each z in the domain of f. You're not supposed to have several possible values of $f(z)$ for a given z. So, if $z_1 = e^0$ and $z_2 = e^{2\pi i}$ stand for the same input then how could $\log z_1$ and $\log z_2$ possibly have two different values? This is clearly a contradiction. How can we clear up this contradiction and still retain the definition that we want for $\log z$? The way out of this seeming contradiction is to regard e^0 and $e^{2\pi i}$ as two different points. We accomplish this by envisioning the domain of the log function as a *Riemann surface.*

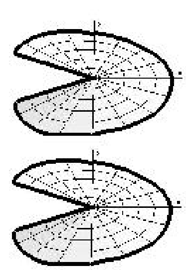
Imagine that the entire complex plane is made of paper and that we slice it along the negative real axis.



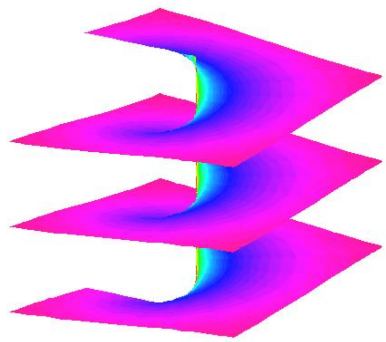
Let's refer to the negative real axis as the *branch cut.* Bend the side above the negative real axis up and the side below the negative real axis down.



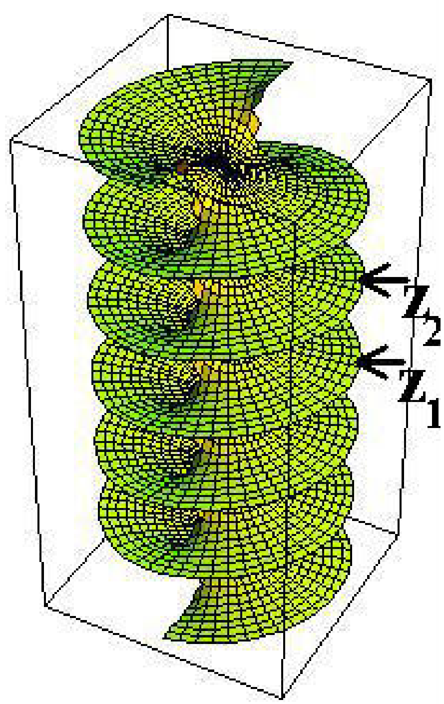
Suppose we have many copies of this piece of paper (that is, many copies of the complex plane) and we have sliced up and folded each in the same way.



Join the surfaces together so that we have created one continuous surface that spirals upwards.

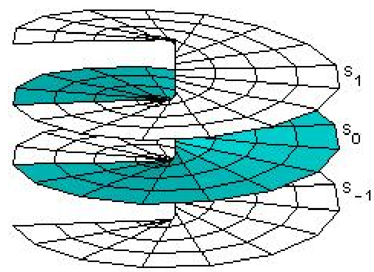


We have just constructed a Riemann Surface. When we have a complex number of the form $re^{i\theta}$ then increasing θ moves you around and up the spiralling surface. $z_1 = e^{i0}$ would be on one level and $z_2 = e^{2\pi i}$ is on the next level up. $e^{4\pi i}$ is directly above $e^{2\pi i}$.



In this way, $e^{2\pi ik}$ is a different point for each integer k.

So, when we integrate $\frac{1}{z}$ around a closed loop C for $-\pi \leq \theta \leq \pi$, the path really isn't a closed loop at all. Rather, we are moving around and up the Riemann surface and the final point is directly above the initial point.



We may now use the formula $\int_{z_1}^{z_2} \frac{1}{z}\, dz = \log z_2 - \log z_1$ to integrate $\oint_C \frac{1}{z}\, dz$:

$$\int_{e^{-\pi i}}^{e^{\pi i}} \frac{1}{z}\, dz = \Big[\log z\Big]_{e^{-\pi i}}^{e^{\pi i}} = \log e^{\pi i} - \log e^{-\pi i} = \pi i - (-\pi i) = 2\pi i$$