Applications of Complex Variables to Fluid Flow

Let $\mathbf{V}$ denote the velocity vector field of a fluid in two dimensions

$$\mathbf{V} = \langle p,\ q\rangle \qquad \text{where } p = p(x,y) \text{ and } q = q(x,y)$$

Note that if we write this in complex number notation, then $\mathbf{V} = p + iq$.

The fluid is *incompressible* if $\nabla \bullet \mathbf{V} = 0$ which implies:

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = 0$$

When will this happen? It will certainly be true if we can find a function $\psi = \psi(x,y)$ with the property:

$$p = \frac{\partial \psi}{\partial y} \qquad q = -\frac{\partial \psi}{\partial x}$$

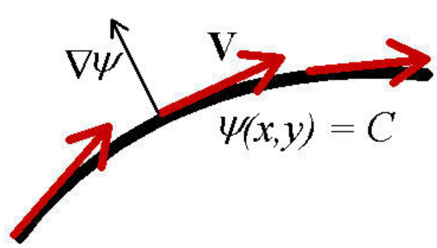
because then,

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

The function ψ is called the *stream function* and it is related to the flow lines (or *streamlines*) that we calculated at the beginning of the course. To see this, first note that the velocity vector field is perpendicular to the gradient of the stream function:

$$\mathbf{V} \bullet \nabla\psi = p\frac{\partial \psi}{\partial x} + q\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0$$

We also know that $\nabla\psi$ is perpendicular to its level sets. That means that $\nabla\psi$ is perpendicular to curves of the form $\psi(x,y) = C$. If $\nabla\psi$ is perpendicular to $\psi(x,y) = C$ and $\mathbf{V}$ is perpendicular to $\nabla\psi$ then $\mathbf{V}$ is tangent to curves of the form $\psi(x,y) = C$ which means that the curves $\psi(x,y) = C$ are the flowlines.



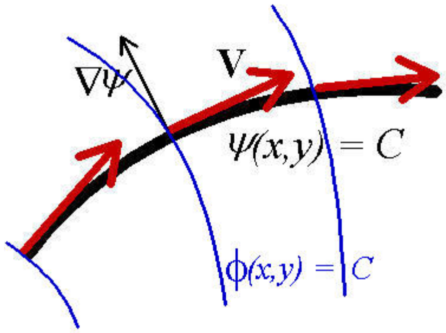
Next, let us suppose that the fluid flow is *irrotational*. This means that $\nabla \times \mathbf{V} = \mathbf{0}$ which implies that:

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$$

If $\nabla \times \mathbf{V} = \mathbf{0}$ then $\mathbf{V}$ is a conservative vector field and therefore must be the gradient of a potential function $\phi(x, y)$

$$\mathbf{V} = \nabla\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$$

Comparing coordinates, we also see that $p = \frac{\partial \phi}{\partial x}$ and $q = \frac{\partial \phi}{\partial y}$. Since $\mathbf{V} = \nabla\phi$ is perpendicular to $\nabla\psi$ and $\nabla\phi$ is perpendicular to its level sets $\phi(x, y) = C$, it must be the case that the *equipotential curves* $\phi(x, y) = C$ must be perpendicular to the streamlines $\psi(x, y) = C$.



Now, switch to complex variable notation. $\mathbf{V} = p + iq$. Define the function $f(z)$ as:

$$f(z) = \phi + i\psi$$

There is an interesting relationship between $f(z)$ and $\mathbf{V}$. First of all, notice that the real and imaginary parts of $f(z)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial \phi}{\partial x} = p = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = q = -\frac{\partial \psi}{\partial x}$$

Therefore, $f(z)$ is an analytic function and it's derivative is given by:

$$f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = p - iq$$

If we compare this to the velocity vector field $\mathbf{V} = p + iq$ we see that $\mathbf{V}$ is the complex conjugate of $f'(z)$

$$\mathbf{V} = \overline{f'(z)}$$