52. $z = 10 - x^2 - y^2, \quad z = 1$

53. $z = x^2 + y^2, \quad x^2 + y^2 = 25, \quad z = 0$

54. $y = x^2 + z^2, \quad 2y = x^2 + z^2 + 4$

55. Find the centroid of the homogeneous solid that is bounded by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and the plane $z = 0$.

56. Find the center of mass of the solid that is bounded by the graphs of $y^2 + z^2 = 16$, $x = 0$, and $x = 5$ if the density at a point P is directly proportional to distance from the yz-plane.

57. Find the moment of inertia about the z-axis of the solid that is bounded above by the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and below by the plane $z = 2$ if the density at a point P is inversely proportional to the square of the distance from the z-axis.

58. Find the moment of inertia about the x-axis of the solid that is bounded by the cone $z = \sqrt{x^2 - y^2}$ and the plane $z = 1$ if the density at a point P is directly proportional to the distance from the z-axis.

In Problems 59–62, convert the point given in spherical coordinates to **(a)** rectangular coordinates and **(b)** cylindrical coordinates.

59. $\left(\frac{2}{3}, \frac{\pi}{2}, \frac{\pi}{6}\right)$ **60.** $\left(5, \frac{5\pi}{4}, \frac{2\pi}{3}\right)$

61. $\left(8, \frac{\pi}{4}, \frac{3\pi}{4}\right)$ **62.** $\left(\frac{1}{3}, \frac{5\pi}{3}, \frac{\pi}{6}\right)$

In Problems 63–66, convert the points given in rectangular coordinates to spherical coordinates.

63. $(-5, -5, 0)$ **64.** $(1, -\sqrt{3}, 1)$

65. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right)$ **66.** $\left(-\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right)$

In Problems 67–70, convert the given equation to spherical coordinates.

67. $x^2 + y^2 + z^2 = 64$ **68.** $x^2 + y^2 + z^2 = 4z$

69. $z^2 = 3x^2 + 3y^2$ **70.** $-x^2 - y^2 + z^2 = 1$

In Problems 71–74, convert the given equation to rectangular coordinates.

71. $\rho = 10$ **72.** $\phi = \pi/3$

73. $\rho = 2 \sec \phi$ **74.** $\rho \sin^2 \phi = \cos \phi$

In Problems 75–82, use triple integrals and spherical coordinates. In Problems 75–78, find the volume of the solid that is bounded by the graphs of the given equations.

75. $z = \sqrt{x^2 + y^2}, \ x^2 + y^2 + z^2 = 9$

76. $x^2 + y^2 + z^2 = 4, \quad y = x, \quad y = \sqrt{3}x, \quad z = 0,$ first octant

77. $z^2 = 3x^2 + 3y^2, \quad x = 0, \quad y = 0, \quad z = 2,$ first octant

78. Inside $x^2 + y^2 + z^2 = 1$ and outside $z^2 = x^2 + y^2$

79. Find the centroid of the homogeneous solid that is bounded by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2z$.

80. Find the center of mass of the solid that is bounded by the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the plane $z = 0$ if the density at a point P is directly proportional to the distance from the xy-plane.

81. Find the mass of the solid that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$ and below by the plane $z = 4$ if the density at a point P is inversely proportional to the distance from the origin. [*Hint*: Express the upper ϕ limit of integration as an inverse cosine.]

82. Find the moment of inertia about the z-axis of the solid that is bounded by the sphere $x^2 + y^2 + z^2 = a^2$ if the density at a point P is directly proportional to the distance from the origin.

9.16 Divergence Theorem

Introduction In Section 9.14 we saw that Stokes' theorem was a three-dimensional generalization of a vector form of Green's theorem. In this section we present a second vector form of Green's theorem and its three-dimensional analogue.

Another Vector Form of Green's Theorem Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a two-dimensional vector field, and let $\mathbf{T} = (dx/ds)\mathbf{i} + (dy/ds)\mathbf{j}$ be a *unit tangent* to a simple closed plane curve C. In (1) of Section 9.14 we saw that $\oint_C (\mathbf{F} \cdot \mathbf{T})\, ds$ can be evaluated by a double integral involving curl $\mathbf{F}$. Similarly, if $\mathbf{n} = (dy/ds)\mathbf{i} - (dx/ds)\mathbf{j}$ is a *unit normal* to C (check $\mathbf{T} \cdot \mathbf{n}$), then $\oint_C (\mathbf{F} \cdot \mathbf{n})\, ds$ can be expressed in terms of a double integral of div $\mathbf{F}$. From Green's theorem,

$$\oint_C (\mathbf{F} \cdot \mathbf{n})\, ds = \oint_C P\, dy - Q\, dx = \iint_R \left[\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\right] dA = \iint_R \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA;$$

that is,

$$\oint_C (\mathbf{F} \cdot \mathbf{n})\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA. \tag{1}$$

The result in (1) is a special case of the **divergence** or **Gauss' theorem**. The following is a generalization of (1) to 3-space:

Theorem 9.16.1 Divergence Theorem

Let D be a closed and bounded region in 3-space with a piecewise-smooth boundary S that is oriented outward. Let $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$ be a vector field for which P, Q, and R are continuous and have continuous first partial derivatives in a region of 3-space containing D. Then

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV. \tag{2}$$

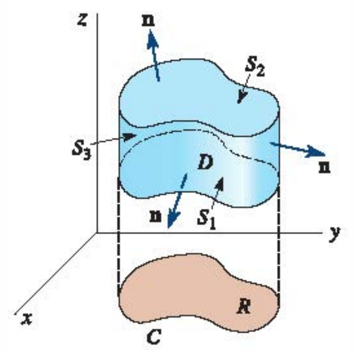


FIGURE 9.16.1 Region D used in proof of Theorem 9.16.1

PARTIAL PROOF: We will prove (2) for the special region D shown in **FIGURE 9.16.1** whose surface S consists of three pieces:

$$\begin{aligned}
&\text{(bottom)} \quad S_1: z = f_1(x, y), \quad (x, y) \text{ in } R\\
&\text{(top)} \quad S_2: z = f_2(x, y), \quad (x, y) \text{ in } R\\
&\text{(side)} \quad S_3: f_1(x, y) \le z \le f_2(x, y), \quad (x, y) \text{ on } C,
\end{aligned}$$

where R is the projection of D onto the xy-plane and C is the boundary of R. Since

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad \text{and} \quad \mathbf{F}\cdot\mathbf{n} = P(\mathbf{i}\cdot\mathbf{n}) + Q(\mathbf{j}\cdot\mathbf{n}) + R(\mathbf{k}\cdot\mathbf{n}),$$

we can write

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_S P(\mathbf{i}\cdot\mathbf{n})\,dS + \iint_S Q(\mathbf{j}\cdot\mathbf{n})\,dS + \iint_S R(\mathbf{k}\cdot\mathbf{n})\,dS$$

and

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D \frac{\partial P}{\partial x}\,dV + \iiint_D \frac{\partial Q}{\partial y}\,dV + \iiint_D \frac{\partial R}{\partial z}\,dV.$$

To prove (2) we need only establish that

$$\iint_S P(\mathbf{i}\cdot\mathbf{n})\,dS = \iiint_D \frac{\partial P}{\partial x}\,dV \tag{3}$$

$$\iint_S Q(\mathbf{j}\cdot\mathbf{n})\,dS = \iiint_D \frac{\partial Q}{\partial y}\,dV \tag{4}$$

$$\iint_S R(\mathbf{k}\cdot\mathbf{n})\,dS = \iiint_D \frac{\partial R}{\partial z}\,dV. \tag{5}$$

Indeed, we shall prove only (5), since the proofs of (3) and (4) follow in a similar manner. Now,

$$\iiint_D \frac{\partial R}{\partial z}\,dV = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z}\,dz\right] dA = \iint_R [R(x, y, f_2(x, y)) - R(x, y, f_1(x, y))]\,dA. \tag{6}$$

Next we write

$$\iint_S R(\mathbf{k}\cdot\mathbf{n})\, dS = \iint_{S_1} R(\mathbf{k}\cdot\mathbf{n})\, dS + \iint_{S_2} R(\mathbf{k}\cdot\mathbf{n})\, dS + \iint_{S_3} R(\mathbf{k}\cdot\mathbf{n})\, dS.$$

On S_1: Since the outward normal points downward, we describe the surface as $g(x, y, z) = f_1(x, y) - z = 0$. Thus,

$$\mathbf{n} = \frac{\dfrac{\partial f_1}{\partial x}\mathbf{i} + \dfrac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial f_1}{\partial x}\right)^2 + \left(\dfrac{\partial f_1}{\partial y}\right)^2}} \quad \text{so that} \quad \mathbf{k}\cdot\mathbf{n} = \frac{-1}{\sqrt{1 + \left(\dfrac{\partial f_1}{\partial x}\right)^2 + \left(\dfrac{\partial f_1}{\partial y}\right)^2}}.$$

From the definition of dS we then have

$$\iint_{S_1} R(\mathbf{k}\cdot\mathbf{n})\, dS = -\iint_R R(x, y, f_1(x, y))\, dA. \tag{7}$$

On S_2: The outward normal points upward, so

$$\mathbf{n} = \frac{-\dfrac{\partial f_2}{\partial x}\mathbf{i} - \dfrac{\partial f_2}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial f_2}{\partial x}\right)^2 + \left(\dfrac{\partial f_2}{\partial y}\right)^2}} \quad \text{so that} \quad \mathbf{k}\cdot\mathbf{n} = \frac{1}{\sqrt{1 + \left(\dfrac{\partial f_2}{\partial x}\right)^2 + \left(\dfrac{\partial f_2}{\partial y}\right)^2}}$$

from which we find

$$\iint_{S_2} R(\mathbf{k}\cdot\mathbf{n})\, dS = \iint_R R(x, y, f_2(x, y))\, dA. \tag{8}$$

On S_3: Since this side is vertical, $\mathbf{k}$ is perpendicular to $\mathbf{n}$. Consequently, $\mathbf{k}\cdot\mathbf{n} = 0$ and

$$\iint_{S_3} R(\mathbf{k}\cdot\mathbf{n})\, dS = 0. \tag{9}$$

Finally, adding (7), (8), and (9), we get

$$\iint_R [R(x, y, f_2(x, y)) - R(x, y, f_1(x, y))]\, dA,$$

which is the same as (6).

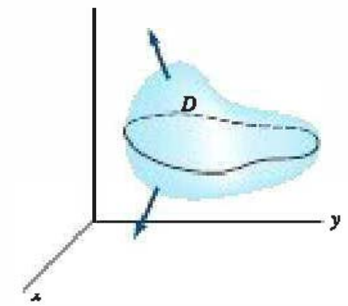


FIGURE 9.16.2 Region D with no vertical side

Although we proved (2) for a special region D that has a vertical side, we note that this type of region is not required in Theorem 9.16.1. A region D with no vertical side is illustrated in **FIGURE 9.16.2**; a region bounded by a sphere or an ellipsoid also does not have a vertical side. The divergence theorem also holds for the region D bounded between two closed surfaces, such as the concentric spheres S_a and S_b shown in **FIGURE 9.16.3**; the boundary surface S of D is the union of S_a and S_b. In this case $\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iiint_D \operatorname{div}\mathbf{F}\, dV$ becomes

$$\iint_{S_b} (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_a} (\mathbf{F}\cdot\mathbf{n})\, dS = \iiint_D \operatorname{div}\mathbf{F}\, dV,$$

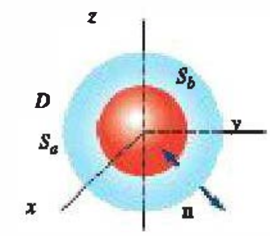


FIGURE 9.16.3 Region D is bounded between two concentric spheres

where **n** points outward from D; that is, **n** points away from the origin on S_b and **n** points toward the origin on S_a.

EXAMPLE 1 Verifying Divergence Theorem

Let D be the region bounded by the hemisphere $x^2 + y^2 + (z-1)^2 = 9$, $1 \le z \le 4$, and the plane $z = 1$. Verify the divergence theorem if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z-1)\mathbf{k}$.

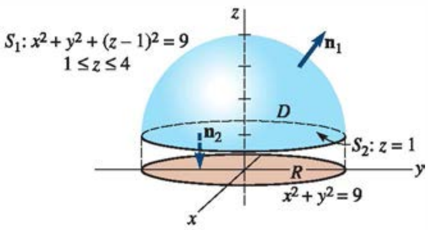


FIGURE 9.16.4 Hemispherical region D in Example 1

SOLUTION The closed region is shown in **FIGURE 9.16.4**.

Triple Integral: Since $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z-1)\mathbf{k}$, we see div $\mathbf{F} = 3$. Hence,

$$\iiint_D \text{div}\,\mathbf{F}\,dV = \iiint_D 3\,dV = 3\iiint_D dV = 54\pi. \tag{10}$$

In the last calculation, we used the fact that $\iiint_D dV$ gives the volume of the hemisphere $(\frac{2}{3}\pi 3^3)$.

Surface Integral: We write $\iint_S = \iint_{S_1} + \iint_{S_2}$, where S_1 is the hemisphere and S_2 is the plane $z = 1$. If S_1 is a level surface of $g(x, y, z) = x^2 + y^2 + (z-1)^2$, then a unit outer normal is

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{x\mathbf{i} + y\mathbf{j} + (z-1)\mathbf{k}}{\sqrt{x^2 + y^2 + (z-1)^2}} = \frac{x}{3}\mathbf{i} + \frac{y}{3}\mathbf{j} + \frac{z-1}{3}\mathbf{k}.$$

Now $$\mathbf{F}\cdot\mathbf{n} = \frac{x^2}{3} + \frac{y^2}{3} + \frac{(z-1)^2}{3} = \frac{1}{3}(x^2 + y^2 + (z-1)^2) = \frac{1}{3}\cdot 9 = 3$$

and so $$\iint_{S_1} (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_R (3)\left(\frac{3}{\sqrt{9 - x^2 - y^2}}\,dA\right)$$

$$= 9\int_0^{2\pi}\int_0^3 (9 - r^2)^{-1/2} r\,dr\,d\theta = 54\pi. \quad \leftarrow \text{polar coordinates}$$

On S_2, we take $\mathbf{n} = -\mathbf{k}$ so that $\mathbf{F}\cdot\mathbf{n} = -z + 1$. But, since $z = 1$, $\iint_{S_2}(-z+1)\,dS = 0$. Hence, we see that $\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = 54\pi + 0 = 54\pi$ agrees with (10). ≡

EXAMPLE 2 Using Divergence Theorem

If $\mathbf{F} = xy\mathbf{i} + y^2z\mathbf{j} + z^3\mathbf{k}$, evaluate $\iint_S(\mathbf{F}\cdot\mathbf{n})\,dS$, where S is the unit cube defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

SOLUTION See Figure 9.13.14 and Problem 38 in Exercises 9.13. Rather than evaluate six surface integrals, we apply the divergence theorem. Since div $\mathbf{F} = \nabla\cdot\mathbf{F} = y + 2yz + 3z^2$, we have from (2)

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iiint_D (y + 2yz + 3z^2)\,dV$$

$$= \int_0^1\int_0^1\int_0^1 (y + 2yz + 3z^2)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (y + 2yz + 3z^2)\,dy\,dz$$

$$= \int_0^1 \left(\frac{y^2}{2} + y^2 z + 3yz^2\right)\Bigg]_0^1 dz$$

$$= \int_0^1 \left(\frac{1}{2} + z + 3z^2\right) dz = \left(\frac{1}{2}z + \frac{1}{2}z^2 + z^3\right)\Bigg]_0^1 = 2. \quad \equiv$$

☐ Physical Interpretation of Divergence

In Section 9.14 we saw that we could express the normal component of the curl of a vector field **F** at a point as a limit involving the circulation of **F**. In view of (2), it is possible to interpret the divergence of **F** at a point as a limit involving the flux of **F**. Recall from Section 9.7 that the flux of the velocity field **F** of a fluid is the rate of fluid flow—that is, the volume of fluid flowing through a surface per unit time. In Section 9.7 we saw that the divergence of **F** is the flux per unit volume. To reinforce this last idea let us suppose $P_0(x_0, y_0, z_0)$ is any point in the fluid and S_r is a small sphere of radius r centered at P_0. See **FIGURE 9.16.5.** If D_r is the sphere S_r and its interior, then the divergence theorem gives

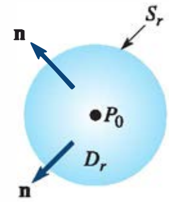


FIGURE 9.16.5 Region D_r in (11)

$$\iint_{S_r} (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_{D_r} \operatorname{div} \mathbf{F}\, dV. \tag{11}$$

If we take the approximation div $\mathbf{F}(P) \approx$ div $\mathbf{F}(P_0)$ at all points $P(x, y, z)$ within the small sphere, then (11) gives

$$\begin{aligned}\iint_{S_r} (\mathbf{F} \cdot \mathbf{n})\, dS &\approx \iiint_{D_r} \operatorname{div} \mathbf{F}(P_0)\, dV \\ &= \operatorname{div} \mathbf{F}(P_0) \iiint_{D_r} dV \qquad (12) \\ &= \operatorname{div} \mathbf{F}(P_0) V_r,\end{aligned}$$

where V_r is the volume $(\frac{4}{3}\pi r^3)$ of the spherical region D_r. By letting $r \to 0$, we see from (12) that the divergence of **F** is the limiting value of the ratio of the flux of **F** to the volume of the spherical region:

$$\operatorname{div} \mathbf{F}(P_0) = \lim_{r \to 0} \frac{1}{V_r} \iint_{S_r} (\mathbf{F} \cdot \mathbf{n})\, dS.$$

Hence, divergence **F** is flux per unit volume.

The divergence theorem is extremely useful in the derivation of some of the famous equations in electricity and magnetism and hydrodynamics. In the discussion that follows we shall consider an example from the study of fluids.

☐ Continuity Equation

At the end of Section 9.7 we mentioned that one interpretation of div **F** was a measure of the rate of change of the density of a fluid at a point. To see why this is so, let us suppose that **F** is a velocity field of a fluid and that $\rho(x, y, z, t)$ is the density of the fluid at a point $P(x, y, z)$ at a time t. Let D be the closed region consisting of a sphere S and its interior. We know from Section 9.15 that the total mass m of the fluid in D is given by $m = \iiint_D \rho(x, y, z, t)\, dV$. The rate at which the mass increases in D is given by

$$\frac{dm}{dt} = \frac{d}{dt} \iiint_D \rho(x, y, z, t)\, dV = \iiint_D \frac{\partial \rho}{\partial t}\, dV. \tag{13}$$

Now from Figure 9.7.3 we saw that the volume of fluid flowing through an element of surface area ΔS per unit time is approximated by $(\mathbf{F} \cdot \mathbf{n})\, \Delta S$. The mass of the fluid flowing through an element of surface area ΔS per unit time is then $(\rho \mathbf{F} \cdot \mathbf{n})\, \Delta S$. If we assume that the change in mass in D is due only to the flow in and out of D, then the *volume of fluid* flowing out of D per unit time is given by (10) of Section 9.13, $\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS$, whereas the *mass of the fluid* flowing out of D per unit time is $\iint_S (\rho\mathbf{F}\cdot\mathbf{n})\,dS$. Hence, an alternative expression for the rate at which the mass increases in D is

$$-\iint_S (\rho\mathbf{F}\cdot\mathbf{n})\,dS. \tag{14}$$

By the divergence theorem, (14) is the same as

$$-\iiint_D \operatorname{div}(\rho\mathbf{F})\,dV. \tag{15}$$

Equating (13) and (15) then yields

$$\iiint_D \frac{\partial\rho}{\partial t}\,dV = -\iiint_D \operatorname{div}(\rho\mathbf{F})\,dV \quad\text{or}\quad \iiint_D \left(\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{F})\right)dV = 0.$$

Since this last result is to hold for every sphere, we obtain the **equation of continuity** for fluid flows:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{F}) = 0. \tag{16}$$

On page 499 we stated that if $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 0$, then a fluid is incompressible. This fact follows immediately from (16). If a fluid is incompressible (such as water), then ρ is constant, so consequently $\nabla\cdot(\rho\mathbf{F}) = \rho\nabla\cdot\mathbf{F}$. But in addition $\partial\rho/\partial t = 0$ and so (16) implies $\nabla\cdot\mathbf{F} = 0$.

9.16 Exercises

Answers to selected odd-numbered problems begin on page ANS-24.

In Problems 1 and 2, verify the divergence theorem.

1. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$; D the region bounded by the unit cube defined by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$
2. $\mathbf{F} = 6xy\mathbf{i} + 4yz\mathbf{j} + xe^{-y}\mathbf{k}$; D the region bounded by the three coordinate planes and the plane $x + y + z = 1$

In Problems 3–14, use the divergence theorem to find the outward flux $\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS$ of the given vector field $\mathbf{F}$.

3. $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$; D the region bounded by the sphere $x^2 + y^2 + z^2 = a^2$
4. $\mathbf{F} = 4x\mathbf{i} + y\mathbf{j} + 4z\mathbf{k}$; D the region bounded by the sphere $x^2 + y^2 + z^2 = 4$
5. $\mathbf{F} = y^2\mathbf{i} + xz^3\mathbf{j} + (z-1)^2\mathbf{k}$; D the region bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 1, z = 5$
6. $\mathbf{F} = x^2\mathbf{i} + 2yz\mathbf{j} + 4z^3\mathbf{k}$; D the region bounded by the parallelepiped defined by $0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3$
7. $\mathbf{F} = y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k}$; D the region bounded within by $z = \sqrt{4 - x^2 - y^2}, x^2 + y^2 = 3, z = 0$
8. $\mathbf{F} = (x^2 + \sin y)\mathbf{i} + z^2\mathbf{j} + xy^3\mathbf{k}$; D the region bounded by $y = x^2, z = 9 - y, z = 0$
9. $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)$; D the region bounded by the concentric spheres $x^2 + y^2 + z^2 = a^2, x^2 + y^2 + z^2 = b^2, b > a$
10. $\mathbf{F} = 2yz\mathbf{i} + x^3\mathbf{j} + xy^2\mathbf{k}$; D the region bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$
11. $\mathbf{F} = 2xz\mathbf{i} + 5y^2\mathbf{j} - z^2\mathbf{k}$; D the region bounded by $z = y$, $z = 4 - y, z = 2 - \frac{1}{2}x^2, x = 0, z = 0$. See **FIGURE 9.16.6**.

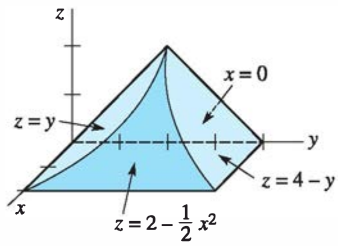


FIGURE 9.16.6 Region D for Problem 11

12. $\mathbf{F} = 15x^2y\mathbf{i} + x^2z\mathbf{j} + y^4\mathbf{k}$; D the region bounded by $x + y = 2$, $z = x + y, z = 3, x = 0, y = 0$
13. $\mathbf{F} = 3x^2y^2\mathbf{i} + y\mathbf{j} - 6zxy^2\mathbf{k}$; D the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$
14. $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + 6\sin x\mathbf{k}$; D the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the planes $z = 2, z = 4$

15. The electric field at a point $P(x, y, z)$ due to a point charge q located at the origin is given by the inverse square field

$$E = q\frac{\mathbf{r}}{\|\mathbf{r}\|^3},$$

where $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Suppose S is a closed surface, S_a is a sphere $x^2 + y^2 + z^2 = a^2$ lying completely within S, and D is the region bounded between S and S_a. See FIGURE 9.16.7. Show that the outward flux of $\mathbf{E}$ for the region D is zero.

(b) Use the result of part (a) to prove **Gauss' law**:

$$\iint_S (\mathbf{E}\cdot\mathbf{n})\,dS = 4\pi q;$$

that is, the outward flux of the electric field $\mathbf{E}$ through *any* closed surface (for which the divergence theorem applies) containing the origin is $4\pi q$.

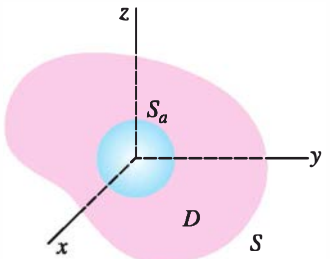


FIGURE 9.16.7 Region D for Problem 15(a)

16. Suppose there is a continuous distribution of charge throughout a closed and bounded region D enclosed by a surface S. Then the natural extension of Gauss' law is given by

$$\iint_S (\mathbf{E}\cdot\mathbf{n})\,dS = \iiint_D 4\pi\rho\,dV,$$

where $\rho(x, y, z)$ is the charge density or charge per unit volume.

(a) Proceed as in the derivation of the continuity equation (16) to show that div $\mathbf{E} = 4\pi\rho$.

(b) Given that $\mathbf{E}$ is an irrotational vector field, show that the potential ϕ for $\mathbf{E}$ satisfies Poisson's equation $\nabla^2\phi = 4\pi\rho$.

In Problems 17–21, assume that S forms the boundary of a closed and bounded region D.

17. If $\mathbf{a}$ is a constant vector, show that $\iint_S (\mathbf{a}\cdot\mathbf{n})\,dS = 0$.

18. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and P, Q, and R have continuous second partial derivatives, prove that

$$\iint_S (\text{curl } \mathbf{F}\cdot\mathbf{n})\,dS = 0.$$

In Problems 19 and 20, assume that f and g are scalar functions with continuous second partial derivatives. Use the divergence theorem to establish **Green's identities**.

19. $$\iint_S (f\nabla g)\cdot\mathbf{n}\,dS = \iiint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV$$

20. $$\iint_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_D (f\nabla^2 g - g\nabla^2 f)\,dV$$

21. If f is a scalar function with continuous first partial derivatives, prove that

$$\iint_S f\mathbf{n}\,dS = \iiint_D \nabla f\,dV.$$

[*Hint*: Use (2) on $f\mathbf{a}$, where $\mathbf{a}$ is a constant vector, and Problem 27 in Exercises 9.7.]

22. The buoyancy force on a floating object is $\mathbf{B} = -\iint_S p\,\mathbf{n}\,dS$, where p is the fluid pressure. The pressure p is related to the density of the fluid $\rho(x, y, z)$ by a law of hydrostatics: $\nabla p = \rho(x, y, z)\mathbf{g}$, where $\mathbf{g}$ is the constant acceleration of gravity. If the weight of the object is $\mathbf{W} = m\mathbf{g}$, use the result of Problem 21 to prove Archimedes' principle, $\mathbf{B} + \mathbf{W} = \mathbf{0}$. See FIGURE 9.16.8.

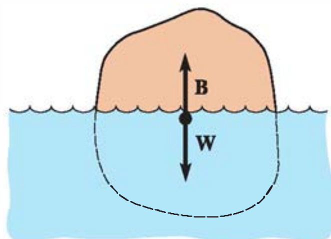


FIGURE 9.16.8 Floating object in Problem 22

9.17 Change of Variables in Multiple Integrals

Introduction In many instances it is either a matter of convenience or of necessity to make a substitution, or change of variable, in a definite integral $\int_a^b f(x)\,dx$ in order to evaluate it. If f is continuous on $[a, b]$, $x = g(u)$ has a continuous derivative, and $dx = g'(u)\,du$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du, \tag{1}$$

If the function g is one-to-one, then it has an inverse and so $c = g^{-1}(a)$ and $d = g^{-1}(b)$.

where the u-limits of integration c and d are defined by $a = g(c)$ and $b = g(d)$. There are three things that bear emphasizing in (1). To change the variable in a definite integral we replace x where it appears in the integrand by $g(u)$, we change the interval of integration $[a, b]$ on the x-axis to the corresponding interval $[c, d]$ on the u-axis, and we replace dx by a function multiple (namely, the derivative of g) of du. If we write $J(u) = dx/du$, then (1) has the form

$$\int_a^b f(x)\, dx = \int_c^d f(g(u))J(u)\, du. \tag{2}$$

For example, using $x = 2 \sin \theta$, $-\pi/2 \le \theta \le \pi/2$, we get

$$\underset{x\text{-limits}}{\int_0^2} \overbrace{\sqrt{4 - x^2}}^{f(x)}\, dx = \underset{\theta\text{-limits}}{\int_0^{\pi/2}} \overbrace{2\cos\theta}^{f(2\sin\theta)}\, \overbrace{(2\cos\theta)}^{J(\theta)}\, d\theta = 4\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi.$$

Double Integrals

Although changing variables in a multiple integral is not as straightforward as the procedure in (1), the basic idea illustrated in (2) carries over. To change variables in a double integral we need two equations such as

$$x = f(u, v), \qquad y = g(u, v). \tag{3}$$

To be analogous with (2), we expect that a change of variables in a double integral would take the form

$$\iint_R F(x, y)\, dA = \iint_S F(f(u, v), g(u, v))J(u, v)\, dA', \tag{4}$$

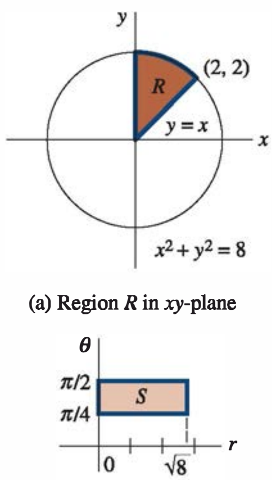
where S is the region in the uv-plane corresponding to the region R in the xy-plane and $J(u, v)$ is some function that depends on the partial derivatives of the equations in (3). The symbol dA' on the right side of (4) represents either $du\, dv$ or $dv\, du$.

In Section 9.11 we briefly discussed how to change a double integral $\iint_R F(x, y)\, dA$ from rectangular coordinates to polar coordinates. Recall that in Example 2 of that section the substitutions

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{5}$$

led to

$$\int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5 + x^2 + y^2}\, dy\, dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{8}} \frac{1}{5 + r^2}\, r\, dr\, d\theta. \tag{6}$$



(a) Region R in xy-plane

(b) Region S in $r\theta$-plane

FIGURE 9.17.1 Region S is used to evaluate (6)

As we see in **FIGURE 9.17.1**, the introduction of polar coordinates changes the original region of integration R in the xy-plane to the more convenient rectangular region of integration S in the $r\theta$-plane. We note, too, that by comparing (4) with (6), we can identify $J(r, \theta) = r$ and $dA' = dr\, d\theta$.

The change-of-variable equations in (3) define a **transformation** or **mapping** T from the uv-plane to the xy-plane. A point (x_0, y_0) in the xy-plane determined from $x_0 = f(u_0, v_0)$, $y_0 = g(u_0, v_0)$ is said to be an **image** of (u_0, v_0).

EXAMPLE 1 Image of a Region

Find the image of the region S shown in **FIGURE 9.17.2(a)** under the transformation $x = u^2 + v^2$, $y = u^2 - v^2$.

SOLUTION We begin by finding the images of the sides of S that we have indicated by S_1, S_2, and S_3.

S_1: On this side $v = 0$ so that $x = u^2$, $y = u^2$. Eliminating u then gives $y = x$. Now imagine moving along the boundary from (1, 0) to (2, 0) (that is, $1 \le u \le 2$). The equations $x = u^2$, $y = u^2$ then indicate that x ranges from $x = 1$ to $x = 4$ and y ranges

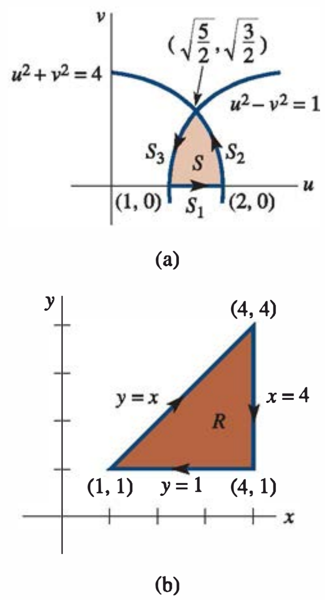


FIGURE 9.17.2 Region R is the image of region S in Example 1