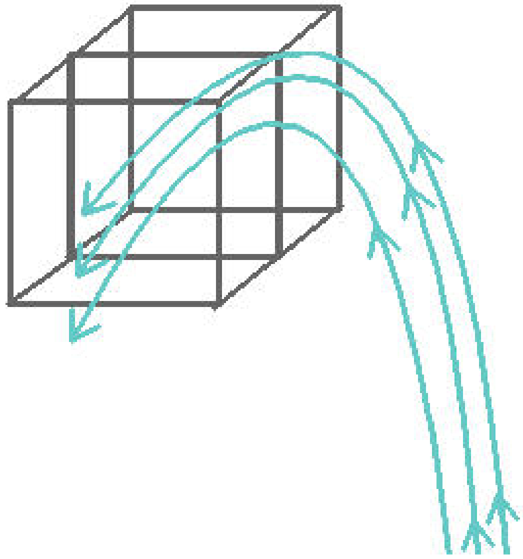
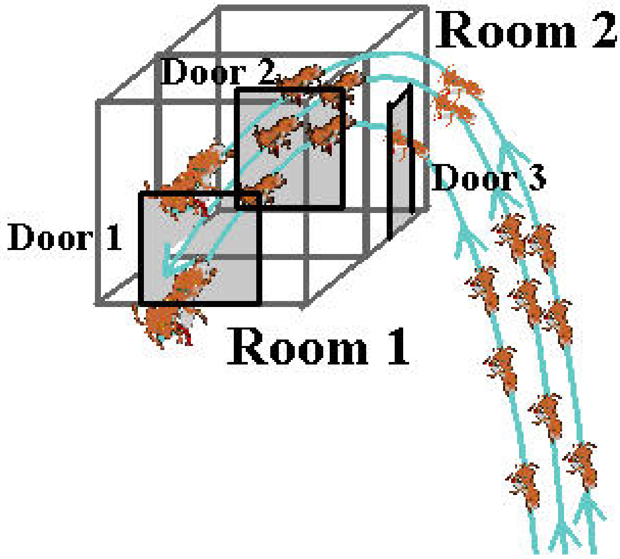
$\vec{\mathbf{F}}$ represents flow through a solid. Divide the solid into two compartments.

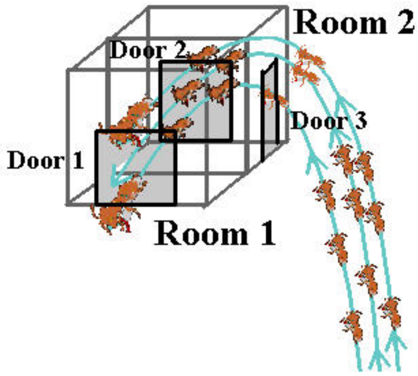
$$\Phi_S = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}} \, dS$$




Room 2
Door 2
Door 3
Door 1
Room 1

Ants enter Room 2 through Door 3 at 4 ants/sec
Ants are passing through Door 2 into Room 1 at 2 ants/sec
Ants are leaving through Door 1 at 1 ant/sec

Therefore, the ant flux is -3 ants/sec



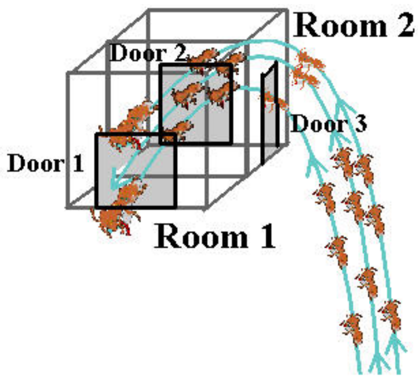
Camera in Room 1 shows an ant flux of -1

$$\text{Flux out of Room 1} = -2 + 1 = -1$$

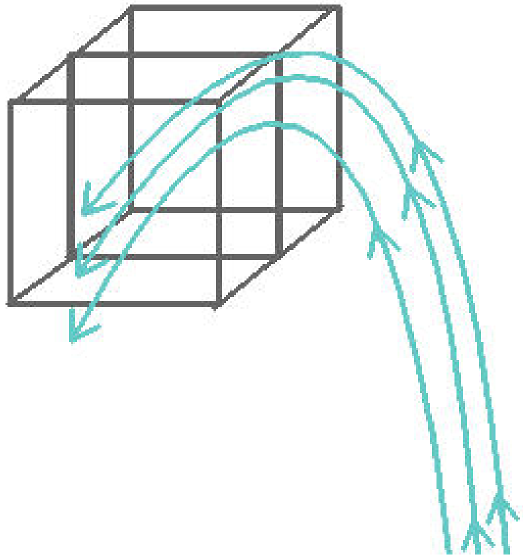
Camera in Room 2 shows an ant flux of -2

$$\text{Flux out of Room 2} = -4 + 2 = -2$$

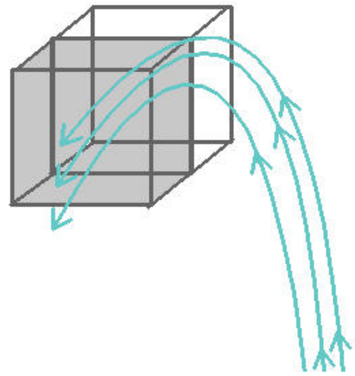
$$\text{Flux (Room 1)} + \text{Flux (Room 2)} = (-2+1) + (-4+2) = -3$$



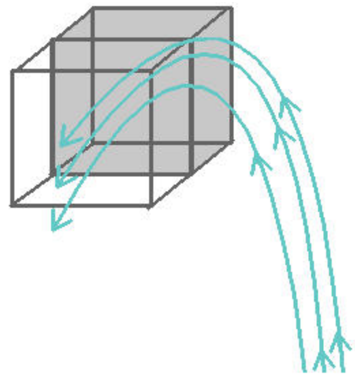
Back to general vector fields



Each compartment has six faces.
first compartment: S_{11}, S_{12}, S_{13}, S_{14}, S_{15} and S_{16}



second compartment: S_{21}, S_{22}, S_{23}, S_{24}, S_{25} and S_{26}.



$$S_{16} = S_{26}$$

The total flux out of the first compartment is:

$$\Phi_{S_{11}} + \Phi_{S_{12}} + \Phi_{S_{13}} + \Phi_{S_{14}} + \Phi_{S_{15}} + \Phi_{S_{16}} = \sum_{j=1}^{6} \Phi_{S_{1j}}$$

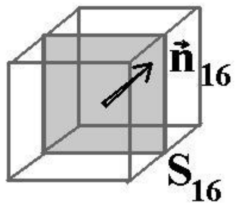
and the total flux out of the second compartment is:

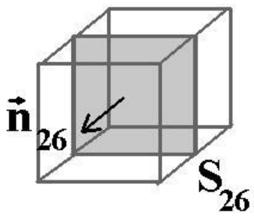
$$\Phi_{S_{21}} + \Phi_{S_{22}} + \Phi_{S_{23}} + \Phi_{S_{24}} + \Phi_{S_{25}} + \Phi_{S_{26}} = \sum_{j=1}^{6} \Phi_{S_{2j}}$$

$$\vec{\mathbf{n}}_{26} = -\vec{\mathbf{n}}_{16}$$

$$\int_{S_{16}} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}_{16}\, dS = \int_{S_{16}} \vec{\mathbf{F}} \bullet (-\vec{\mathbf{n}}_{26})\, dS = -\int_{S_{26}} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}_{26}\, ds$$

$$\Phi_{S_{16}} = -\Phi_{S_{26}}$$



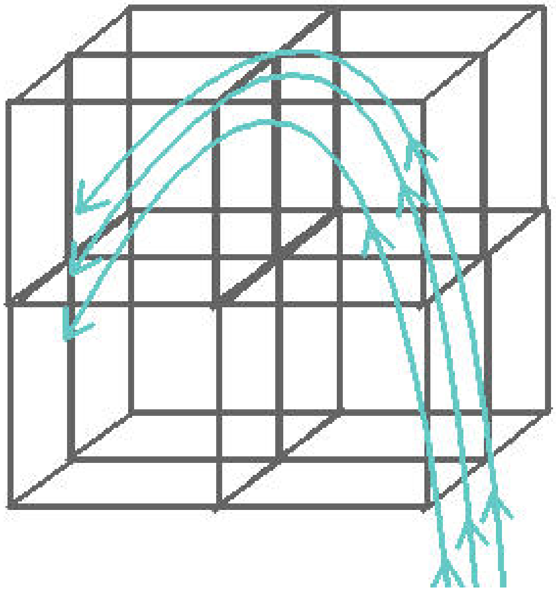


$$
\begin{aligned}
\sum_{j=1}^{6}\Phi_{S_{1j}} + \sum_{j=1}^{6}\Phi_{S_{2j}}
&= \left(\sum_{j=1}^{5}\Phi_{S_{1j}} + \Phi_{S_{16}}\right) + \left(\sum_{j=1}^{5}\Phi_{S_{2j}} + \Phi_{S_{26}}\right) \\
&= \left(\sum_{j=1}^{5}\Phi_{S_{1j}} - \Phi_{S_{26}}\right) + \left(\sum_{j=1}^{5}\Phi_{S_{2j}} + \Phi_{S_{26}}\right) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{5}\Phi_{S_{ij}}
\end{aligned}
$$

$$\sum_{i=1}^{2}\sum_{j=1}^{6}\Phi_{S_{ij}} = \sum\sum_{\text{exterior faces}}\Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

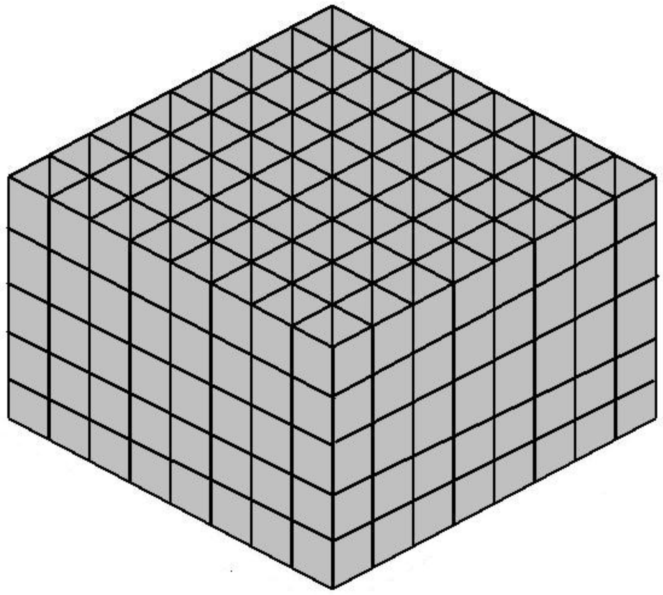
where S is the entire surface surrounding the two combined compartments.

Generalize to more interior compartments.



$$\sum_{i=1}^{8}\sum_{j=1}^{6}\Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

n compartments.



$$\sum_{i=1}^{n}\sum_{j=1}^{6}\Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

Divergence $\nabla \bullet \vec{\mathbf{F}}$ is the flux per unit volume as the volume shrinks to 0.

$$\sum_{j=1}^{6} \Phi_{S_{ij}} \approx \nabla \bullet \vec{\mathbf{F}}\, \mathrm{vol}(V_i)$$

where V_i is the interior of the i^{th} compartment.

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where V_i is the interior of the i^{th} compartment.

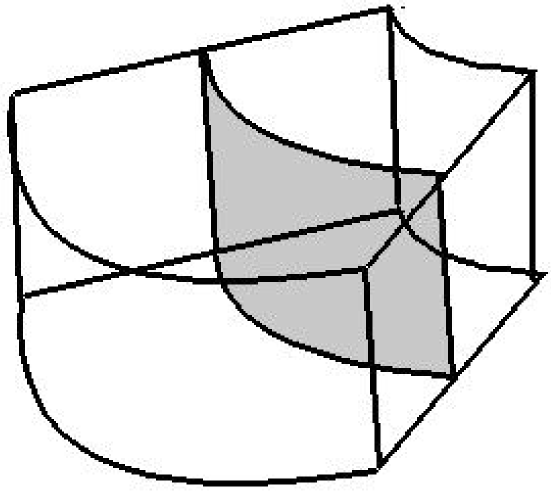
$$\sum_{i=1}^{n} \nabla \bullet \vec{\mathbf{F}}\,\mathrm{vol}(V_i) \approx \sum_{i=1}^{n}\sum_{j=1}^{6} \Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\,dS$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\nabla\bullet\vec{\mathbf{F}}\,\mathrm{vol}(V_i)=\iint_S\vec{\mathbf{F}}\bullet\vec{\mathbf{n}}\,dS$$

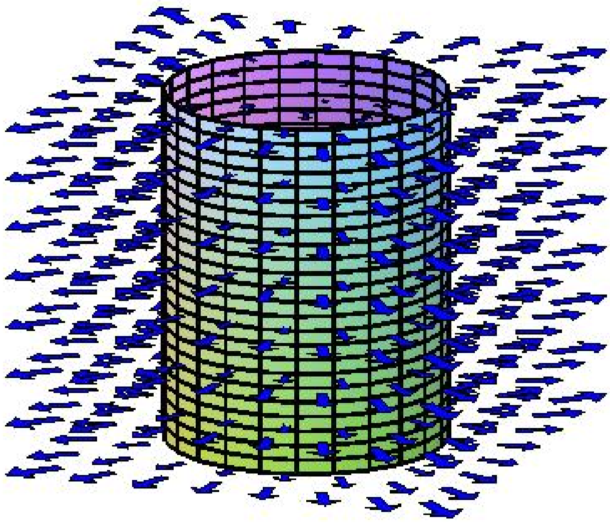
$$\iiint_V\nabla\bullet\vec{\mathbf{F}}\,dV=\iint_S\vec{\mathbf{F}}\bullet\vec{\mathbf{n}}\,dS$$

Divergence Theorem.

We could have used solids with different shapes.



Example: Let $\vec{\mathbf{F}} = \langle x,\ y,\ 0\rangle$, let S be the surface surrounding the cylinder V described by the inequalities: $x^2 + y^2 \leq 4$ and $0 \leq z \leq 5$.



$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iiint_V \operatorname{div} \vec{\mathbf{F}}\, dV$$

Let's try this and compare. Let T denote the top of the cylinder and let B denote the bottom of the cylinder.
$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ is the same as:

$$\iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_{\text{Side}} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

However,

$$\iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_T \langle x, y, 0\rangle \bullet \langle 0, 0, 1\rangle\, dS = 0$$

$$\iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_B \langle x, y, 0\rangle \bullet \langle 0, 0, -1\rangle\, dS = 0$$

$$\iint_{\text{Side}} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\text{Side}} \langle x, y, 0\rangle \bullet \frac{\langle x, y, 0\rangle\, dS}{\sqrt{x^2 + y^2}}$$

$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iint_{\text{Side}} \sqrt{x^2+y^2}\, dS \\
&= \iint_{\text{Side}} 2\, dS \\
&= 2 \cdot \text{Area(Side)} \\
&= 40\pi
\end{aligned}$$

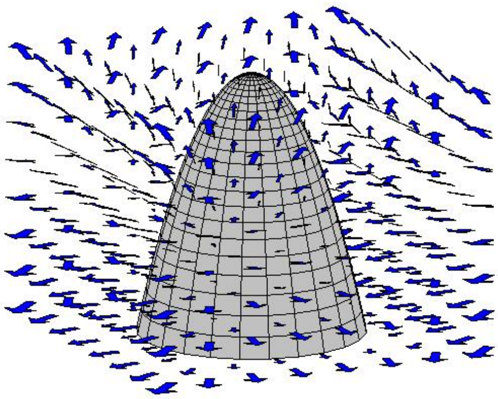
$$\iiint_V \operatorname{div} \vec{\mathbf{F}}\, dV = \iiint_V 2\, dV = 2 \cdot \mathrm{Vol(V)} = 2 \cdot \pi \cdot 2^2 \cdot 5 = 40\pi$$

Divergence Theorem - Conditions

$\nabla \bullet \vec{\mathbf{F}}$ must exist at all points in the interior of the solid

S must be a *closed* surface.

Example: Let V be the region bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy plane.
Let S be the surface that completely surrounds V.
Let Ω be the parabolic portion of S
Let B be the disk at the bottom.
Let $\vec{\mathbf{F}} = \langle x^2, y^2, z \rangle$. Calculate $\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$.



The surface Ω is not a closed surface but S is.

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

We can use the Divergence Theorem for $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ because $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iiint_V \nabla \bullet \vec{\mathbf{F}}\, dV$ and therefore,

$$\iiint_V \nabla \bullet \vec{\mathbf{F}}\, dV = \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

$$\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iiint_{V} \nabla \bullet \vec{\mathbf{F}}\, dV - \iint_{B} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

$$\iint_{B} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{B} \langle x^2,\ y^2,\ 0\rangle \bullet \langle 0,\ 0,\ -1\rangle\, dx\, dy = 0$$

$$
\begin{aligned}
\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iiint_{V} \nabla \bullet \vec{\mathbf{F}}\, dV - \iint_{B} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \\
&= \iiint_{V} (2x + 2y + 1) dV - 0 \\
&= \int_0^1 \int_0^{1-r^2} \int_0^{2\pi} (2r\cos\theta + 2r\sin\theta + 1)\, r\, d\theta\, dz\, dr + 0 \\
&= \frac{\pi}{2}
\end{aligned}
$$