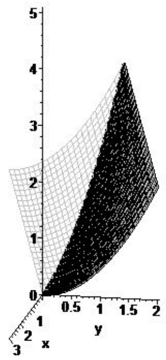
Assignments - Spring 2023

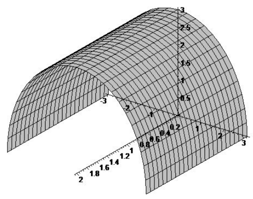
Assignment 1. *Surface Area*

Hand in the following problems:

1. Let Q be the region in the xy plane bounded by the lines $x = 0$, $y = 2$ and $3y - 2x = 0$. Find the surface area of the portion of the surface $z = x + \frac{y^2}{2}$ that is directly over Q.

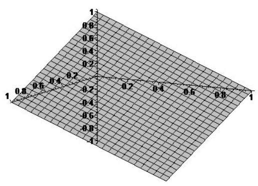


2. Find the surface area of the portion of the cylinder $y^2 + z^2 = 9$ above the rectangle in the xy plane where $0 \leq x \leq 2$ and $-3 \leq y \leq 3$



3. Find the surface area of the surface S defined by the equation:

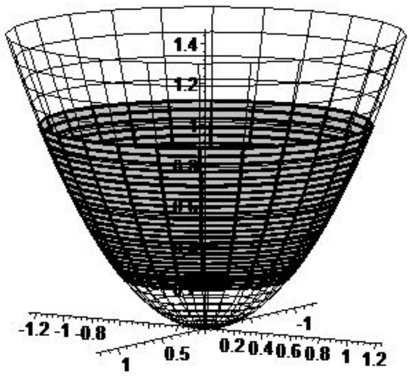
$$\langle x,\ y,\ z\rangle = \langle u,\ v,\ 1 - u - v\rangle \qquad \text{where } 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 1$$



Continue to next page for problems 4 and 5

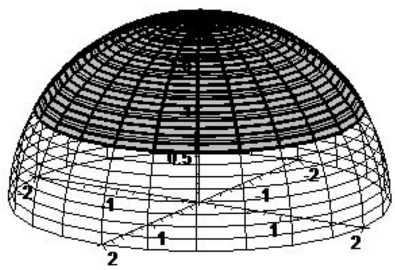
4. Find the surface area of the paraboloid S defined by the equation:

$$\langle x,\ y,\ z\rangle = \left\langle u\cos v,\ u\sin v,\ u^2\right\rangle \qquad \text{where } \frac{1}{2} \le u \le 1 \text{ and } 0 \le v \le 2\pi$$



5. The following equation describes a portion of a sphere of radius 2

$$\langle x,\ y,\ z\rangle = \langle\, 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\,\rangle \qquad \text{where } 0 \le \theta \le 2\pi \text{ and } 0 \le \phi \le \frac{\pi}{3}$$



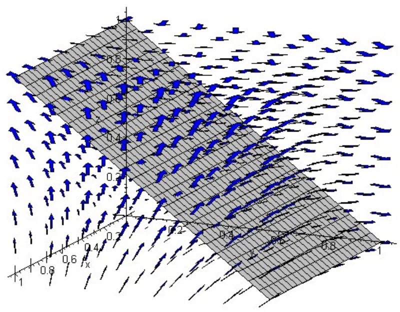
a. Find the surface area of this portion of the sphere.

b. If electric charge is distributed over this surface and has a charge density of $\delta = \frac{1}{2}z$ coulombs/m^2, then the total amount of charge on this surface is $\iint_S \frac{1}{2}z\,dS$. Calculate this integral.

Assignment 2. *Surface Integrals*

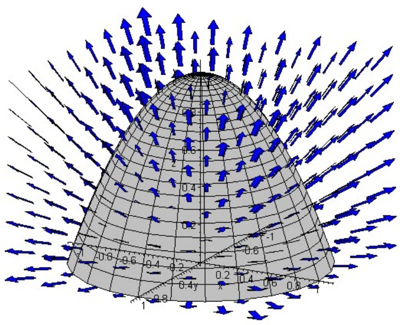
Hand in the following problems:

1. Let S_P be the planar surface described by the equation $\vec{\mathbf{r}} = \langle x,\ y,\ 1-y\rangle$ for $0 \le x \le 1$ and $0 \le y \le 1$. Let $\vec{\mathbf{F}} = \langle z,\ y,\ x\rangle$.



Calculate the surface integral $\iint_{S_P} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$.

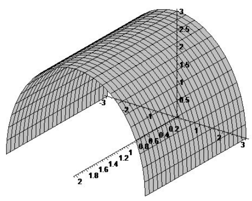
2. Let Ω be the parabolic surface described by the equation $\vec{\mathbf{r}} = \langle u\cos\theta,\ u\sin\theta,\ 1-u^2\rangle$ for $0 \le u \le 1$ and $0 \le \theta \le 2\pi$. Let $\vec{\mathbf{F}} = \langle 2x,\ 2y,\ 2z\rangle$.



Calculate the surface integral $\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$.

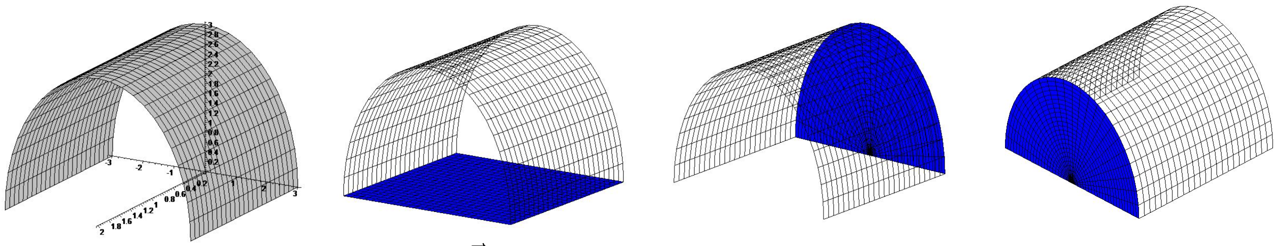
3. Let S be the sphere described by the equation $\vec{\mathbf{r}} = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$ for $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. Let $\vec{\mathbf{F}} = \langle -y,\ x,\ 0\rangle$. Calculate the surface integral $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$.

4. Let Ω the portion of the cylinder $y^2 + z^2 = 9$ above the rectangle in the xy plane where $0 \le x \le 2$ and $-3 \le y \le 3\}$. You found the surface area of this surface in Assignment 1.



Let $\vec{\mathbf{F}} = \langle x,\ y,\ y+z\rangle$ Calculate the surface integral $\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$.

5. Again, let Ω be the surface described in Problem 4 of this assignment and let $\vec{\mathbf{F}}$ be the vector field given by the equation: $\vec{\mathbf{F}} = \langle x,\ y,\ y+z \rangle$. Let V be the solid bounded from above by Ω and below by the xy plane for $= -3 \leq y \leq 3$. Let S be the *closed surface* that completely surrounds V. S includes not only Ω, but also two semicircular sides at $x = 0$ and $x = 2$ and a rectangular base on the xy plane.



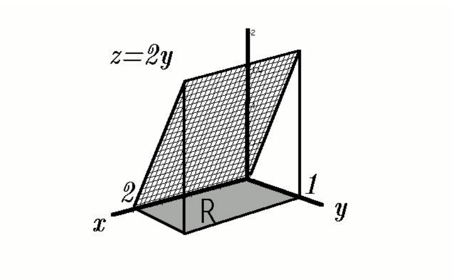
Calculate the surface integral $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ by combining the surface integrals over all four boundaries.

Assignment 3. *Divergence Theorem*

Hand in the following problems:

1. Let T be the three dimensional solid bounded from above by the paraboloid described by the equation $z = 1 - x^2 - y^2$ and from below by $z = 0$. Let S be the closed surface that completely surrounds T. Let $\vec{\mathbf{F}} = \langle 2x,\ 2y,\ 2z \rangle$. Use the Divergence Theorem to calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$.

2. Let R be the rectangular region with $0 \le x \le 2$ and $0 \le y \le 1$. Let V be the three dimensional region that lies above R but below the plane $z = 2y$. Let S be the closed surface surrounding V.



Let $\vec{\mathbf{F}} = \langle x+y,\ y+z,\ z-x \rangle$. Use the Divergence Theorem to evaluate the surface integral $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\ dS$.

3. Let T be the three dimensional solid bounded from above by the half cylinder described by the equation $z = \sqrt{9 - y^2}$ and from below the xy plane for $0 \le x \le 2$ and $-3 \le y \le 3$. Let S be the closed surface that completely surrounds T. Let $\vec{\mathbf{F}} = \langle x,\ y,\ y + z \rangle$.

You calculated $\iint_s \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ on Assignment 2 already. This time, calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ using the Divergence Theorem.

4. Let T be the quarter sphere solid that is inside $x^2 + y^2 + z^2 = 1$ for $x \ge 0$ and $y \ge 0$. Let S be the closed surface that completely surrounds T. Let $\vec{\mathbf{F}} = (3z + y)\vec{\mathbf{k}}$. Use the Divergence Theorem to calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$

5. Let $\vec{\mathbf{F}} = \left(x^2 + y^2\right)\vec{\mathbf{j}}$. Let S be the closed surface surrounding the rectangular box with vertices:

$(0,0,0)$	$(1,0,0)$	$(1,2,0)$	$(0,2,0)$
$(0,0,3)$	$(1,0,3)$	$(1,2,3)$	$(0,2,3)$

Evaluate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ using the Divergence Theorem

Assignment 4. *Divergence and Curl*

Hand in the following problems:

1. Find div $\vec{\mathbf{F}}$, given that:

(a) $\vec{\mathbf{F}} = e^{xy}\,\vec{\mathbf{i}} + \sin\, xy\,\vec{\mathbf{j}} + \cos^2\ zx\,\vec{\mathbf{k}}$.

(b) $\vec{\mathbf{F}} = \nabla\phi$, where $\phi = 3x^2y^3z$.

2. Find **curl** $\vec{\mathbf{F}}$ if $\vec{\mathbf{F}} = z^2x\vec{\mathbf{i}} + y^2z\vec{\mathbf{j}} - z^2y\vec{\mathbf{k}}$.

3. Given $\vec{\mathbf{F}}(x, y, z) = x^2y\vec{\mathbf{i}} + z\vec{\mathbf{j}} - (x + y - z)\vec{\mathbf{k}}$, find

(a) $\nabla\bullet\vec{\mathbf{F}}$ (b) $\nabla\times\vec{\mathbf{F}}$ (c) $\nabla(\nabla\bullet\vec{\mathbf{F}})$

4. Let $\phi = \phi(x, y, z)$ be some scalar-valued function. Let $\vec{\mathbf{F}} = \langle F_1,\ F_2,\ F_3\rangle$ be some vector field where each coordinate may depend on x, y and z. Classify each of the following expressions as *vector*, *scalar* or *meaningless.*

$$\nabla\bullet(\nabla\phi) \qquad \nabla\times(\nabla\bullet\vec{\mathbf{F}}) \qquad \nabla\bullet(\nabla^2\phi) \qquad \vec{\mathbf{F}}\bullet\nabla\phi \qquad \vec{\mathbf{F}}\times\nabla\phi$$

5. Let $\phi(x, y) = x^2y^4 + xy$. Let $\vec{\mathbf{F}} = \langle x,\ -z,\ y\rangle$ Calculate each of the following:

a. $\nabla\phi$ **b.** $\nabla^2\phi$ **c.** $\nabla\bullet\vec{\mathbf{F}}$ **d.** $\nabla\times\vec{\mathbf{F}}$

Assignment 5. *Line Integrals*

Hand in the following problems:

1. The following equation describes a parabolic path C:

$$\vec{\mathbf{r}}(t) = \langle t,\ 2,\ 1-t^2 \rangle \qquad -1 \le t \le 1$$

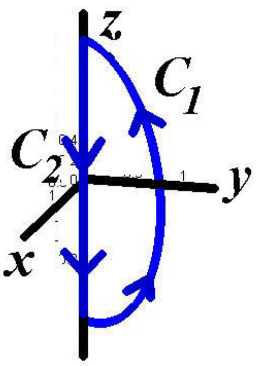
If $\vec{\mathbf{F}} = \langle 3z,\ x+y+z,\ 2x \rangle$, calculate the line integral $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$

2. The following equation describes a semiellipse C_1:

$$\vec{\mathbf{r}}(t) = \langle \cos t,\ \cos t,\ \sin t \rangle \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

If $\vec{\mathbf{F}} = \langle 0,\ 2z,\ 4y+1 \rangle$, calculate the line integral $\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$

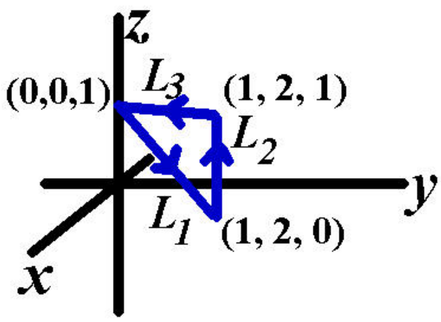
3. Let C_1 be the same curve defined in problem 2 and again let $\vec{\mathbf{F}} = \langle 0,\ 2z,\ 4y+1 \rangle$. Let C_2 be the straight line segment from $(0,\ 0,\ 1)$ to $(0,\ 0,\ -1)$. Together, path C_1 followed by path C_2 form a closed loop C.



Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ by combining $\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ with $\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$.

4. Let L_1 be the straight line segment from $(0,\ 0,\ 1)$ to $(1,\ 2,\ 0)$ and let $\vec{\mathbf{F}} = \langle z+2x,\ 2y,\ x \rangle$. Calculate the line integral $\int_{L_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$

5. Let L_1 and $\vec{\mathbf{F}}$ be the same as defined in problem 4. Let L_2 be the straight line segment from $(1,\ 2,\ 0)$ to $(1,\ 2,\ 1)$. Let L_3 be the straight line segment from $(1,\ 2,\ 1)$ to $(0,\ 0,\ 1)$. Path L_1 followed by path L_2 and then by path L_3 forms a closed triangular loop C. Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ by combining $\int_{L_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$, $\int_{L_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ and $\int_{L_3} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$.



Assignment 6. *Fundamental Theorem for Line Integrals*

1. For each vector field $\vec{\mathbf{F}}$, find a scalar valued function $\phi = \phi(x, y)$ so that $\vec{\mathbf{F}} = \nabla\phi$.

a.
$$\vec{\mathbf{F}} = \left\langle \frac{y^2}{2x},\ y\ln x \right\rangle$$

b.
$$\vec{\mathbf{F}} = \left\langle y + \frac{1}{x},\ x \right\rangle$$

2. Let $\vec{\mathbf{F}} = \left\langle \frac{y^2}{2x},\ y\ln x \right\rangle$

Calculate $\int_\Gamma \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ where Γ is the straight line segment connecting $(1, 1)$ to $(2, 2)$.

3. Let H be the segment of the hyperbola $x^2 - y^2 = 1$ connecting $(1, 0)$ to $(2, \sqrt{3})$. Evaluate the integral:

$$\int_H \left(y + \frac{1}{x}\right) dx + x\, dy$$

4. Let $\vec{\mathbf{F}} = \langle 2y\sqrt{y} + 4,\ 3x\sqrt{y} \rangle$

a. Find a scalar-valued function $\phi = \phi(x, y)$ so that $\vec{\mathbf{F}} = \nabla\phi$

b. Calculate the line integral $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ where C is any path connecting $(1,\ 1)$ to $\left(\frac{1}{2},\ 4\right)$

5. Let $\vec{\mathbf{F}} = \langle e^x y,\ e^x + z,\ y \rangle$.

Let L be the straight line path from $(1,\ 0,\ 0)$ to $(0,\ \pi,\ \pi)$. Calculate the following line integral:

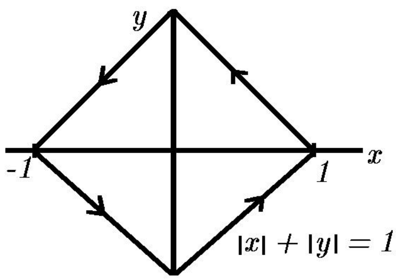
$$\int_L \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

Assignment 7. *Stokes' Theorem*

1. Let C be the circle of radius 1 around the origin in the yz plane. Let $\vec{\mathbf{F}} = \langle 0,\ y + 4z,\ 6y + z \rangle$.

Use Stokes' Theorem to calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$.

2. Let C be the square in the xy plane with equation $|x| + |y| = 1$.



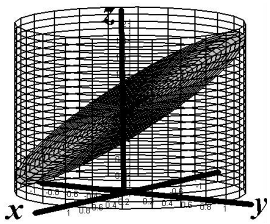
Let $\vec{\mathbf{F}} = -y\vec{\mathbf{i}} + x\vec{\mathbf{j}}$. Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ in two ways:

a) Add up the integrals over all four line segments of C.

b) Use Stokes' Theorem.

3. Let $\vec{\mathbf{F}} = -2y\vec{\mathbf{i}} + x\vec{\mathbf{j}} + x\vec{\mathbf{k}}$

Let C be the intersection of the plane $z = 1 + y$ with the cylinder $x^2 + y^2 = 1$ and orient C counterclockwise as viewed from above.



Use Stokes' Theorem to calculate the line integral $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$.

4. Use Stokes' Theorem to find the closed loop line integral in Assignment 5 Problem 3.

5. Use Stokes' Theorem to find the closed loop line integral in Assignment 5 Problem 5.

Assignment 8. *Stokes' Theorem and Green's Theorem*

1. Let R be the region in the first and second quadrants of the xy plane bounded by the lines $y = x$, $y = -x$ and the circle $x^2 + y^2 = 4$. Let C be the boundary of this region, traversed in a counterclockwise direction. Let $\vec{\mathbf{v}} = x^2 y \vec{\mathbf{i}} + xy^2 \vec{\mathbf{j}}$. Use Green's Theorem to evaluate $\oint_C \vec{\mathbf{v}} \bullet d\vec{\mathbf{r}}$

2. Let C denote the triangular path connecting $(0, 0, 0)$ to $(1, 0, 0)$ to $(0, 1, 0)$ and back to $(0, 0, 0)$. C forms the boundary of a triangular region S_{xy} in the xy plane. Show how Green's Theorem can be used to calculate:

$$\oint_C xy\,dx + \frac{1}{2}\left(x^2 + x + y\right)\,dy$$

3. Let R be the region in the yz plane below the curve $z = \sqrt{y}$ and above the y-axis between $y = 0$ and $y = 2$. Let C be the curve surrounding this region R. Assume that C is traversed in the counterclockwise direction as viewed from the positive x-axis.

Let $\vec{\mathbf{F}} = \frac{1}{2} z^2 \vec{\mathbf{j}} + y \vec{\mathbf{k}}$. Use Stokes' Theorem to evaluate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$

4. Let S be the portion of the surface $z = 1 + y^2$ bounded by the vertical planes $x = 0$, $x = 1$, $y = 0$ and $y = 1$. Let C be the curve that forms the boundary of S. You may assume that C is traversed in the counterclockwise direction. Define $\vec{\mathbf{F}}$ as follows:

$$\vec{\mathbf{F}} = z\vec{\mathbf{i}} + 2x\vec{\mathbf{j}} + 3y\vec{\mathbf{k}}$$

Evaluate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ using Stokes' Theorem.

5. Let H be the hemisphere described by the equation

$$z = \sqrt{1 - x^2 - y^2}$$

Let $\vec{\mathbf{F}} = z\vec{\mathbf{i}}$. Use any legitimate method to calculate the surface integral:

$$\iint_H (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\,dS$$

Assignment 9. *Fourier Series*

1. Classify each of the following functions as *even*, *odd* or *neither even nor odd*:

a. $f(x) = x^3 + x$

b. $f(x) = \sin^2 x \cos x$

c. $f(x) = x - |x|$

d. $f(x) = \begin{cases} x-x^2 & \text{for } x \geq 0 \\ x^2+x & \text{for } x \leq 0 \end{cases}$

Problems 2 - 5. A Fourier series for a function $f(x)$ on the interval $-L \leq x \leq L$ has the form:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

For each of the following problems, calculate the Fourier series for the given function $f(x)$ and the specified value of L. Also, graph the sum of the first three non-zero terms of the series. Please note that for Problem 4, you may leave your answer in terms of $\cos\frac{n\pi}{2}$.

2. $$f(x) = \begin{cases} \pi - x & \text{for } x \geq 0 \\ -\pi - x & \text{for } x < 0 \end{cases} \qquad L = \pi$$

3. $$f(x) = |x| - x \qquad L = \pi$$

4. $$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x \leq -\frac{\pi}{2} \\ -1 & \text{for } -\frac{\pi}{2} \leq x \leq 0 \\ 1 & \text{for } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \leq x \leq \pi \end{cases} \qquad L = \pi$$

5. $$f(x) = \begin{cases} x - x^2 & \text{for } 0 \leq x \leq 1 \\ x^2 + x & \text{for } -1 \leq x \leq 0 \end{cases} \qquad L = 1$$

Assignment 10. *Ordinary Differential Equations and Complex Number*

This is a review assignment if you have already taken MA 345. If you haven't taken MA 345, then watch the videos on differential equations that have been provided.

For problems 1 - 3, find the general solution $y = y(t)$ for each of the following differential equations.

1.
$$\frac{d^2y}{dt^2} - 16y = 0$$

2.
$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = 0$$

3.
$$\frac{d^2y}{dt^2} + \pi^2 y = 0$$

4. Use Euler's formula to write each of the following in terms of sine and cosine. Simplify as much as possible.

a. $e^{8\pi i}$ **b.** $e^{i\frac{\pi}{2}}$ **c.** $e^{-1+3\pi i}$ **d.** $e^{ix} + e^{-ix}$

5. For each of the following exponential equations, use Euler's formula and find the most general expression for θ that makes the equation true.

a. $e^{2i\theta} = 1$ **b.** $e^{-i\theta} = -1$

Assignment 11. *Partial Differential Equations*

1. If we substitute $u = X(x)T(t)$ into the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

we obtain

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

where λ is called the *separation constant.*

If u satisfies the boundary conditions $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(\pi, t) = 0$, find the possible values of λ that lead to nontrivial solutions for u.

2. If we substitute $u(x,t) = X(x)Y(y)$ into the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ we end up with:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2$$

where λ is a constant. For one of the following sets of boundary conditions, $\lambda = 0$ leads to a nontrivial solution. Which one?

a. $u(0,\ y) = 0$ and $u(\pi,\ y) = 0$

b. $\frac{\partial u}{\partial x}(0,\ y) = 0$ and $\frac{\partial u}{\partial x}(\pi,\ y) = 0$

c. $\frac{\partial u}{\partial x}(0,\ y) = 0$ and $u(\pi,\ y) = 0$

3. If we find the Fourier series solution $u = u(x,t)$ for the equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

where u is subject to the boundary conditions $u(0,t) = 0$ and $u(\pi, t) = 0$ then we obtain:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2 t} \sin nx$$

If the initial condition is $u(x,0) = \sin x \cos x$ then calculate the coefficients b_n.

Hint: If you use an appropriate trigonometric identity for $\sin x \cos x$*, you won't have to find any antiderivatives at all to see the answer.*

4. Suppose $u = u(x,t)$ is the solution to the partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

and u also satisfies the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$ for all t as well as the initial conditions $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = x - x^2$ for all x in the interval from $x = 0$ to $x = 1$. Find the Fourier series solution for $u(x,t)$.