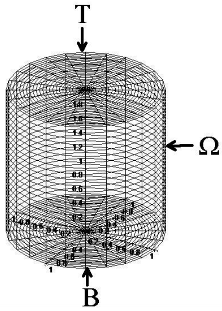
1. *(16 points)* Let V be the three dimensional region inside the cylinder $x^2 + y^2 = 1$ for $0 \le z \le 2$. Let S be the closed surface surrounding V.

S consists of 3 surfaces, Ω, B and T shown below.



Let $\vec{\mathbf{F}} = \langle xz,\ yz,\ 0\rangle$. Calculate the surface integral $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ by adding up the surface integrals over Ω, B and T. Show all work.

$\vec{\mathbf{F}}$ has no vertical component so $\iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = 0$ and $\iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = 0$

On Ω, $\vec{\mathbf{r}} = \langle \cos\theta,\ \sin\theta,\ z\rangle$ so $\vec{\mathbf{n}}\, dS = \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial z}\, d\theta\, dz = \langle \cos\theta,\ \sin\theta\ 0\rangle\, d\theta\, dz$

$$\begin{aligned}\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \int_0^2 \int_0^{2\pi} \langle z\cos\theta,\ z\sin\theta,\ 0\rangle \bullet \langle \cos\theta,\ \sin\theta,\ 0\rangle\, d\theta\, dz \\ &= \int_0^2 \int_0^{2\pi} z\, d\theta\, dz = 4\pi\end{aligned}$$

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2. *(16 points)* Let S and $\vec{\mathbf{F}}$ be defined exactly as in problem 1. Calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ using the Divergence Theorem. Show all work.

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iiint_V 2z\, dV = \int_0^2 \int_0^1 \int_0^{2\pi} 2zr\, d\theta\, dr\, dz = 4\pi$$

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3. *(16 points)* Let $\vec{\mathbf{F}} = \langle 0,\ 2x + 2y,\ 0\rangle$.

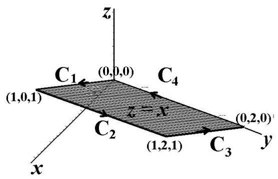
Let C_1 be the straight line path from $(0,\ 0,\ 0)$ to $(1,\ 0,\ 1)$

Let C_2 be the straight line path from $(1,\ 0,\ 1)$ to $(1,\ 2,\ 1)$

Let C_3 be the straight line path from $(1,\ 2,\ 1)$ to $(0,\ 2,\ 0)$

Let C_4 be the straight line path from $(0,\ 2,\ 0)$ to $(0,\ 0,\ 0)$

Together, these paths form a closed rectangular loop C. All points on C lie on the plane $z = x$.



Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ by adding up the line integrals over C_1, C_2, C_3 and C_4. Show all work.

$dy = 0$ on C_1 and C_3, so $\int_{C_1}(2x+2y)\,dy = \int_{c_3}(2x+2y)\,dy = 0$
On C_2, $\int_{C_2}(2x+2y)\,dy = \int_0^2(2+2y)\,dy = 8$
On C_4, $\int_{C_4}(2x+2y)\,dy = \int_2^0(0+2y)\,dy = -4$
Adding up all four integrals:

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0 + 8 + 0 - 4 = 4$$

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4. *(16 points)* Let $\vec{\mathbf{F}}$ and C be defined exactly as in Problem 3. Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ using Stokes' Theorem. Show all work.
$\nabla \times \vec{\mathbf{F}} = \langle 0,\ 0,\ 2\rangle$.
$\vec{\mathbf{r}} = \langle x,\ y,\ x\rangle$ so $\vec{\mathbf{n}}\,dS = \frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y}\,dx\,dy = \langle -1,\ 0,\ 1\rangle\,dx\,dy$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^2\int_0^1 \langle 0,\ 0,\ 2\rangle \bullet \langle -1,\ 0,\ 1\rangle\,dx\,dy = \int_0^2\int_0^1 2\,dx\,dy = 4$$

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5. *(16 points)* Find the Fourier series for $f(x) = 4 - \frac{1}{2}x$ on the interval $-\pi \le x \le \pi$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(4 - \frac{1}{2}x\right)\,dx = 8$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(4 - \frac{1}{2}x\right)\cos nx\,dx = 0 \text{ for } n \ge 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(4 - \frac{1}{2}x\right)\sin nx\,dx = \frac{(-1)^n}{n}$$

$$f(x) = 4 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

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6. *(20 points)* Let $u(x,t)$ be the solution of $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$, where $u(x,t)$ satisfies the boundary conditions $u(0,\ t) = 0$ and $u(1,\ t) = 0$. Obtain the solution in the form of a Fourier series by first looking for solutions of the form $X(x)T(t)$. There are no initial conditions given, so your Fourier series answer will have coefficients that remain undetermined.

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\sin n\pi x$$