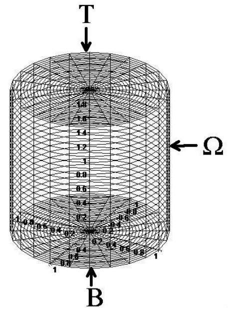
MA 441 Spring 2022 Final Exam - B Dr. E. Jacobs

1. *(16 points)* Let V be the three dimensional region inside the cylinder $x^2 + y^2 = 1$ for $0 \le z \le 2$. Let S be the closed surface surrounding V.

S consists of 3 surfaces, Ω, B and T shown below.



Let $\vec{\mathbf{F}} = \langle -y,\ x,\ \pi z \rangle$. Calculate the surface integral $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ by adding up the surface integrals over Ω, B and T. Show all work.

$\vec{\mathbf{F}}$ has no vertical component on B so $\iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = 0$

$\iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_T \langle -y,\ x,\ 2\pi \rangle \bullet \langle 0,\ 0,\ 1 \rangle\, dx\, dy = \iint_T 2\pi\, dx\, dy = 2\pi^2$

On Ω, $\vec{\mathbf{r}} = \langle \cos\theta,\ \sin\theta,\ z \rangle$ so $\vec{\mathbf{n}}\, dS = \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial z}\, d\theta\, dz = \langle \cos\theta,\ \sin\theta,\ 0 \rangle\, d\theta\, dz$

$$\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \int_0^2 \int_0^{2\pi} \langle -\sin\theta,\ \cos\theta,\ \pi z \rangle \bullet \langle \cos\theta,\ \sin\theta,\ 0 \rangle\, d\theta\, dz = 0$$

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = 0 + 2\pi^2 + 0 = 2\pi^2$$

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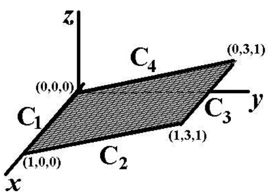
2. *(16 points)* Let S and $\vec{\mathbf{F}}$ be defined exactly as in problem 1. Calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ using the Divergence Theorem. Show all work.

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iiint_V \pi\, dV = \int_0^2 \int_0^1 \int_0^{2\pi} \pi\, r\, d\theta\, dr\, dz = 2\pi^2$$

Let $\vec{\mathbf{F}} = \langle 8x - 4xy,\ 0,\ 0 \rangle$.
Let C_1 be the straight line path from $(0,\ 0,\ 0)$ to $(1,\ 0,\ 0)$
Let C_2 be the straight line path from $(1,\ 0,\ 0)$ to $(1,\ 3,\ 1)$
Let C_3 be the straight line path from $(1,\ 3,\ 1)$ to $(0,\ 3,\ 1)$
Let C_4 be the straight line path from $(0,\ 3,\ 1)$ to $(0,\ 0,\ 0)$
Together, these paths form a closed rectangular loop C. All points on C lie on the plane $z = \frac{1}{3}y$.



Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ by adding up the line integrals over C_1, C_2, C_3 and C_4. Show all work.
$dx = 0$ on C_2 and C_4, so $\int_{C_2} (8x - 4xy)\, dx = \int_{c_4} (8x - 4xy)\, dx = 0$
On C_1, $\int_{C_1} (8x - 4xy)\, dx = \int_0^1 8x\, dx = 4$
On C_3, $\int_{C_3} (8x - 4xy)\, dx = \int_1^0 (8x - 12x)\, dx = \int_0^1 4x\, dx = 2$
Adding up all four integrals:

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0 + 4 + 2 + 0 = 6$$

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4. *(16 points)* Let $\vec{\mathbf{F}}$ and C be defined exactly as in Problem 3. Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ using Stokes' Theorem. Show all work.
$\nabla \times \vec{\mathbf{F}} = \langle 0,\ 0,\ 4x \rangle$.
$\vec{\mathbf{r}} = \langle x,\ y,\ y \rangle$ so $\vec{\mathbf{n}}\, dS = \frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y}\, dx\, dy = \langle 0,\ -1/3,\ 1 \rangle\, dx\, dy$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^3 \int_0^1 \langle 0,\ 0,\ 4x \rangle \bullet \langle 0,\ -1/3,\ 1 \rangle\, dx\, dy = \int_0^3 \int_0^1 4x\, dx\, dy = 6$$

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5. *(16 points)* Find the Fourier series for the following function for $-\pi \le x \le \pi$.

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for} \quad 0 \le x \le \pi \\ -\frac{1}{2}x & \text{for} \quad -\pi \le x < 0 \end{cases}$$

This is an even function so $b_n = 0$ and $a_n = \frac{2}{\pi} \int_0^\pi \frac{1}{2} x \cos nx\, dx = \frac{1}{\pi} \int_0^\pi x \cos nx\, dx$

$$a_0 = \frac{1}{\pi} \int_0^\pi x\, dx = \frac{\pi}{2} \qquad a_n = \frac{1}{\pi} \int_0^\pi x \cos nx\, dx = \frac{1}{\pi n^2} \left((-1)^n - 1 \right)$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2} \left((-1)^n - 1 \right) \cos nx$$

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6. *(20 points)* Let $u(x,t)$ be the solution of $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$, where $u(x,t)$ satisfies the boundary conditions $u(0,\ t) = 0$ and $u(1,\ t) = 0$. Obtain the solution in the form of a Fourier series by first looking for solutions of the form $X(x)T(t)$. There are no initial conditions given, so your Fourier series answer will have coefficients that remain undetermined.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin n\pi x$$