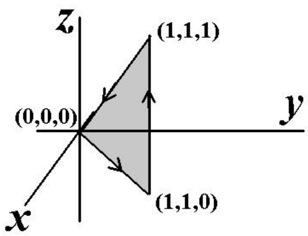
1. The following three points all lie in the plane $y = x$:

$$(0,\ 0,\ 0) \qquad (1,\ 1,\ 0) \qquad (1,\ 1,\ 1)$$

Let C be the closed triangular loop consisting of the straight line segments from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 0)$ and then from $(1,\ 1,\ 0)$ to $(1,\ 1,\ 1)$ and finally from $(1,\ 1,\ 1)$ back to $(0,\ 0,\ 0)$.



Let $\vec{\mathbf{F}} = \langle x,\ -yz,\ y^2 \rangle$.

a. Calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ by calculating the integrals over all three sides of the triangle and combining them.

Let L_1 be the line from the origin to $(1,\ 1,\ 0)$.
Let L_2 be the line from $(1,\ 1,\ 0)$ to $(1,\ 1,\ 1)$
Let L_3 be the line from $(1,\ 1,\ 1)$ to the origin.

$$\int_{L_1} x\,dx - yz\,dy + y^2\,dz = \int_0^1 x\,dx = \frac{1}{2}$$

$$\int_{L_2} x\,dx - yz\,dy + y^2\,dz = \int_0^1 1\,dz = 1$$

$$\int_{L_3} x\,dx - yz\,dy + y^2\,dz = \int_1^0 x\,dx = -\frac{1}{2}$$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \frac{1}{2} + 1 - \frac{1}{2} = 1$$

b. Use Stokes' Theorem to calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$.

The surface can be described by the equation $\vec{\mathbf{r}} = \langle y,\ y,\ z \rangle$

$$\begin{aligned}\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \iint_{\mathcal{D}} \nabla \times \vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial y} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} \right) dA \\ &= \int_0^1 \int_0^y \langle 3y,\ 0,\ 0 \rangle \bullet \langle 1,\ -1,\ 0 \rangle \, dz\, dy \\ &= \int_0^1 \int_0^y 3y \, dz\, dy = 1\end{aligned}$$

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2. Let $\vec{\mathbf{F}} = \left\langle 2xy^2 + z^2,\ 2x^2y,\ 2xz \right\rangle$

a. Show that $\vec{\mathbf{F}}$ is a conservative vector field

All we need to do is demonstrate that $\nabla \times \vec{\mathbf{F}}$ is the zero vector.

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + z^2 & 2x^2y & 2xz \end{vmatrix} = (0-0)\vec{\mathbf{i}} - (2z - 2z)\vec{\mathbf{k}} + (4xy - 4xy)\vec{\mathbf{k}} = \vec{\mathbf{0}}$$

b. Find a scalar-valued function ϕ so that $\vec{\mathbf{F}} = \nabla\phi$

$$\phi(x, y, z) = xz^2 + x^2y^2$$

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3. Again, let $\vec{\mathbf{F}} = \left\langle 2xy^2 + z^2,\ 2x^2y,\ 2xz^2 \right\rangle$.

Use the Fundamental Theorem for Line Integrals to calculate $\int_{(0,0,0)}^{(2,1,1)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$.

$$\int_{(0,0,0)}^{(2,1,1)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(2,1,1) - \phi(0,0,0) = 6 - 0 = 6$$

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4. Let R be the region in the xy plane below $y = 1 - x^2$ but above the x-axis. Let C be the closed loop surrounding R. Let $\vec{\mathbf{F}} = \left\langle 6xy,\ 6x + 3x^2 \right\rangle$. Use Green's Theorem to calculate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \int_{-1}^1 \int_0^{1-x^2} 6\, dy\, dx = 8$$

5. Find the Fourier series for the following function for $-1 \leq x \leq 1$:

$$f(x) = \begin{cases} 1 + x & \text{for} \quad 0 \leq x \leq 1 \\ -1 + x & \text{for} - 1 \leq x < 0 \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - 2(-1)^n) \sin n\pi x$$

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6. Find the Fourier series for the function $f(x) = \pi - |x|$ on the interval $-\pi \leq x \leq \pi$

$f(x)$ is an even function, so $b_n = 0$ for all n.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)\, dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x)\, dx = \pi$$

$$\text{For } n \geq 1, \quad a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx\, dx = \frac{2}{n^2\pi} (1 - (-1)^n)$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} (1 - (-1)^n) \cos nx$$