$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$L = 1$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$

$$a_n = \int_{-1}^{1} f(x) \cos n\pi x \, dx$$

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx$$

Find the Fourier series for $f(x) = \sin^2 \pi x$ for $L = 1$

Find the Fourier series for $f(x) = \sin^2 \pi x$ for $L = 1$

$$\sin^2 \pi x = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$

$$a_n = \int_{-1}^{1} \sin^2 \pi x \cos n\pi x \, dx$$

$$b_n = \int_{-1}^{1} \sin^2 \pi x \sin n\pi x \, dx$$

$f(x) = \sin^2(\pi x)$ is an *even function*

$$
\begin{aligned}
f(-x) &= \sin^2(-\pi x) \\
&= (\sin(-\pi x))^2 \\
&= (-\sin(\pi x))^2 \\
&= \sin^2(\pi x) \\
&= f(x)
\end{aligned}
$$

$$\sin^2 \pi x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$

$$\begin{aligned} a_n &= \int_{-1}^{1} \sin^2 \pi x \cos n\pi x \, dx \\ &= 2\int_{0}^{1} \sin^2 \pi x \cos n\pi x \, dx \end{aligned}$$

$$b_n = \int_{-1}^{1} \sin^2 \pi x \sin n\pi x \, dx = 0$$

$$a_n = 2\int_0^1 \sin^2 \pi x \cos n\pi x \, dx$$

Use Euler's formula on $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ we get:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Use Euler's formula on $e^{i(\alpha-\beta)} = e^{i\alpha}e^{-i\beta}$ we get:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

If we add, we get:

$$\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2\cos\alpha\cos\beta$$

If we subtract, we get:

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$$

$$a_n = 2\int_0^1 \sin^2 \pi x \cos n\pi x \, dx$$

So,

$$a_0 = 2\int_0^1 \sin^2 \pi x \, dx$$

Substitute $\alpha = \pi x$ and $\beta = \pi x$ into

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta$$

$$1 - \cos 2\pi x = 2 \sin^2 \pi x$$

$$
\begin{aligned}
a_0 &= 2\int_0^1 \sin^2 \pi x \, dx \\
&= \int_0^1 (1 - \cos 2\pi x) \, dx \\
&= \left[ x - \frac{\sin 2\pi x}{2\pi} \right]_0^1 \\
&= 1
\end{aligned}
$$

Calculate for $n \geq 1$:

$$a_n = 2\int_0^1 \sin^2 \pi x \cos n\pi x \, dx$$

We already know that $2\sin^2 \pi x = 1 - \cos 2\pi x$

$$a_n = \int_0^1 (1 - \cos 2\pi x)(\cos n\pi x) \, dx$$

$$
\begin{aligned}
a_n &= \int_0^1 (1 - \cos 2\pi x)(\cos n\pi x)\,dx \\
&= \int_0^1 \cos n\pi x\,dx - \int_0^1 \cos 2\pi x \cos n\pi x\,dx \\
&= 0 - \int_0^1 \cos 2\pi x \cos n\pi x\,dx
\end{aligned}
$$

Substitute $\alpha = 2\pi x$ and $\beta = n\pi x$ into

$$\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2\cos\alpha\cos\beta$$

$$\cos(2\pi x - n\pi x) + \cos(2\pi x + n\pi x) = 2\cos 2\pi x \cos n\pi x$$

$$\cos((2-n)\pi x) + \cos((2+n)\pi x) = 2\cos 2\pi x \cos n\pi x$$

$$
\begin{aligned}
a_n &= -\int_0^1 \cos 2\pi x \cos n\pi x \, dx \\
&= -\int_0^1 \frac{1}{2}(\cos((2-n)\pi x) + \cos((2+n)\pi x)) \\
&= -\frac{1}{2} \cdot \left[ \frac{\sin((2-n)\pi x)}{(2-n)\pi} + \frac{\sin((2+n)\pi x)}{(2+n)\pi} \right]_0^1 \\
&= 0
\end{aligned}
$$

(assuming $n \neq 2$)

If $n = 2$ then:

$$\begin{aligned}
a_n &= -\int_0^1 \cos 2\pi x \cos n\pi x \, dx \\
&= -\int_0^1 \frac{1}{2}(\cos((2-n)\pi x) + \cos((2+n)\pi x)) \, dx \\
&= -\int_0^1 \frac{1}{2}(1 + \cos 4\pi x) \, dx \\
a_2 &= -\frac{1}{2}
\end{aligned}$$

$$\sin^2 \pi x = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{2} \qquad a_2 = -\frac{1}{2} \qquad a_n = 0 \text{ (for all other } n\text{'s)}$$

$$\begin{aligned}\sin^2 \pi x &= \frac{1}{2}a_0 + a_1 \cos 1\pi x + a_2 \cos 2\pi x + a_3 \cos 3\pi x + \cdots \\ &= \frac{1}{2} \cdot 1 + 0 \cdot \cos 1\pi x - \frac{1}{2} \cos 2\pi x + 0 \cdot \cos 3\pi x + \cdots \\ &= \frac{1}{2} - \frac{1}{2} \cos 2\pi x\end{aligned}$$

$$\sin^2 \pi x = \frac{1}{2} - \frac{1}{2}\cos 2\pi x$$

More generally,

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$