$$f(x) = \sum_{n=0}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x) \quad \text{where } \omega_n = \frac{n\pi}{L}$$

Take the limit as $L \to \infty$

$$f(x) = \int_0^{\infty} (A(\omega) \cos \omega x + B(\omega) \sin \omega x) \, d\omega$$

$$
\begin{aligned}
f(x) &= \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right) d\omega \\
&= \int_0^\infty \left(A(\omega)\left(\frac{e^{i\omega x}+e^{-i\omega x}}{2}\right) + B(\omega)\left(\frac{e^{i\omega x}-e^{-i\omega x}}{2i}\right)\right) d\omega \\
&= \int_0^\infty \frac{A(\omega)-iB(\omega)}{2} e^{i\omega x} d\omega + \int_0^\infty \frac{A(\omega)+iB(\omega)}{2} e^{-i\omega x} d\omega \\
&= \int_0^\infty \frac{A(\omega)-iB(\omega)}{2} e^{i\omega x} d\omega + \int_{-\infty}^0 \frac{A(\omega)-iB(\omega)}{2} e^{i\omega x} d\omega \\
&= \int_{-\infty}^\infty \frac{A(\omega)-iB(\omega)}{2} e^{i\omega x} d\omega
\end{aligned}
$$

$$\text{Let } s = -\omega$$

$$\begin{aligned}\int_0^{\infty} \frac{A(\omega)+iB(\omega)}{2} e^{-i\omega x} d\omega &= \int_{-\infty}^{0} \frac{A(-s)+iB(-s)}{2} e^{isx}\, ds \\ &= \int_{-\infty}^{0} \frac{A(s)-iB(s)}{2} e^{isx}\, ds \\ &= \int_{-\infty}^{0} \frac{A(\omega)-iB(\omega)}{2} e^{i\omega x}\, d\omega\end{aligned}$$

$$f(x) = \int_{-\infty}^{\infty} \frac{A(\omega) - iB(\omega)}{2} e^{i\omega x} d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx$$

$$\begin{aligned} \frac{A(\omega) - iB(\omega)}{2} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)(\cos \omega x - i \sin \omega x) \, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \end{aligned}$$

$$f(x) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \right) e^{i\omega x} \, d\omega$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \right) e^{i\omega x} \, d\omega$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \right) e^{i\omega x}\, d\omega$$

Define the Fourier transform $\mathcal{F}(f(x))$ to be:

$$\mathcal{F}(f(x)) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \right) e^{i\omega x} \, d\omega$$

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This last integral is the inverse Fourier transform of $\hat{f}(\omega)$.