$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx = \mathcal{F}(f)$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega = \mathcal{F}^{-1}(\hat{f})$$

$$G(x,t) = \int_a^b f(x,t,\omega)\, d\omega$$

$$G(x+h,t) = \int_a^b f(x+h,t,\omega)\, d\omega$$

$$G(x,t) = \int_a^b f(x,t,\omega)\, d\omega$$

$$G(x+h,t) = \int_a^b f(x+h,t,\omega)\,d\omega$$

$$G(x,t) = \int_a^b f(x,t,\omega)\,dt$$

$$\frac{G(x+h,t) - G(x,t)}{h} = \int_a^b \frac{f(x+h,t,\omega) - f(x,t,\omega)}{h}\,d\omega$$

$$\frac{G(x+h,t)-G(x,t)}{h} = \int_a^b \frac{f(x+h,t,\omega)-f(x,t,\omega)}{h}\,d\omega$$

Take the limit as $h \to 0$

$$\frac{\partial G}{\partial x} = \int_a^b \frac{\partial f}{\partial x}\,d\omega$$

$$\frac{\partial}{\partial x}\int_a^b f(x,t,\omega)\,d\omega = \int_a^b \frac{\partial f}{\partial x}(x,t,\omega)\,d\omega$$

Let $u = u(x, t)$ be the solution of the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\hat{u}(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x} \, dx$$

$$u(x, t) = \int_{-\infty}^{\infty} \hat{u}(t, \omega) e^{i\omega x} \, d\omega$$

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(t,\omega) e^{i\omega x}\, d\omega$$

$$\frac{\partial u}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial \hat{u}}{\partial t}(t,\omega) e^{i\omega x}\, d\omega$$

$$\frac{\partial^2 u}{\partial t^2} = \int_{-\infty}^{\infty} \frac{\partial^2 \hat{u}}{\partial t^2}(t,\omega) e^{i\omega x}\, d\omega$$

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(t,\omega) e^{i\omega x} \, d\omega$$

$$\frac{\partial u}{\partial x} = \int_{-\infty}^{\infty} i\omega \hat{u}(t,\omega) e^{i\omega x} \, d\omega$$

$$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} -\omega^2 \hat{u}(t,\omega) e^{i\omega x} \, d\omega$$

$$\frac{\partial^2 u}{\partial t^2} = \int_{-\infty}^{\infty} \frac{\partial^2 \hat{u}}{\partial t^2}(t,\omega) e^{i\omega x}\, d\omega = \mathcal{F}^{-1}\left(\frac{\partial^2 \hat{u}}{\partial t^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} -\omega^2 \hat{u}(t,\omega) e^{i\omega x}\, d\omega = \mathcal{F}^{-1}\left(-\omega^2 \hat{u}\right)$$

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t^2}\right) = \frac{\partial^2 \hat{u}}{\partial t^2}$$

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) = -\omega^2 \hat{u}$$

Solve the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = 0$
Take the Fourier transform of both sides of the equation:

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t^2}\right) = \mathcal{F}\left(c^2 \frac{\partial^2 u}{\partial x^2}\right)$$

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t^2}\right) = c^2 \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t^2}\right) = c^2 \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2 \omega^2 \hat{u}$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2 \omega^2 \hat{u} = 0$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2 \omega^2 \hat{u} = 0$$

$$\hat{u}(\omega,\ t) = A(\omega) \cos c\omega t + B(\omega) \sin c\omega t$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2 \omega^2 \hat{u} = 0$$

$$\hat{u}(\omega,\ t) = A(\omega) \cos c\omega t + B(\omega) \sin c\omega t$$

$$\hat{u}(\omega,\ 0) = A(\omega)$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2\omega^2\hat{u} = 0$$

$$\hat{u}(\omega,\ t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t$$

$$\hat{u}(\omega,\ 0) = A(\omega)$$

$$\hat{u}(\omega,\ t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega t}\,dx$$

$$\hat{u}(\omega,\ 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)e^{-i\omega x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$\hat{u}(\omega,\ t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t$$

$$\frac{\partial \hat{u}}{\partial t}(\omega,\ t) = -c\omega A(\omega)\sin c\omega t + c\omega B(\omega)\cos c\omega t$$

$$\frac{\partial \hat{u}}{\partial t}(\omega,\ 0) = c\omega B(\omega)$$

$$\hat{u}(\omega,\ t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t$$

$$\frac{\partial \hat{u}}{\partial t}(\omega,\ t) = -c\omega A(\omega)\sin c\omega t + c\omega B(\omega)\cos c\omega t$$

$$\frac{\partial \hat{u}}{\partial t}(\omega,\ 0) = c\omega B(\omega)$$

$$\hat{u}(\omega,\ t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx$$

$$\frac{\partial \hat{u}}{\partial t}(\omega,\ t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -i\omega\frac{\partial u}{\partial t}(x,t)e^{-i\omega x}\,dx$$

$$\frac{\partial \hat{u}}{\partial t}(\omega,\ 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 0\,dx = 0 \quad \text{so } B(\omega) = 0$$

$$\hat{u}(\omega,\ t) = A(\omega)\cos c\omega t \quad \text{where } A(\omega) = \mathcal{F}(f(x)) = \hat{f}(\omega)$$

$$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\omega,\ t))$$

$$\hat{u}(\omega,\ t) = A(\omega)\cos c\omega t \quad \text{where } A(\omega) = \mathcal{F}(f(x)) = \hat{f}(\omega)$$

$$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\omega,\ t))$$

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\omega,\ t)e^{i\omega x}\, d\omega = \int_{-\infty}^{\infty} \hat{f}(\omega)\cos c\omega t e^{i\omega x}\, d\omega$$

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \hat{f}(\omega) \cos c\omega t e^{i\omega x}\, d\omega \\
&= \int_{-\infty}^{\infty} \hat{f}(\omega) \frac{1}{2} \left(e^{ic\omega t} + e^{-ic\omega t} \right) e^{i\omega x}\, d\omega \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i(x+ct)}\, d\omega + \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i(x-ct)}\, d\omega
\end{aligned}
$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega$$

$$f(x + ct) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega(x+ct)} \, d\omega$$

$$f(x - ct) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega(x-ct)} \, d\omega$$

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \hat{f}(\omega) \cos c\omega t e^{i\omega x}\, d\omega \\
&= \int_{-\infty}^{\infty} \hat{f}(\omega) \frac{1}{2} \left(e^{ic\omega t} + e^{-ic\omega t} \right) e^{i\omega x}\, d\omega \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i(x+ct)}\, d\omega + \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i(x-ct)}\, d\omega \\
&= \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct)
\end{aligned}
$$