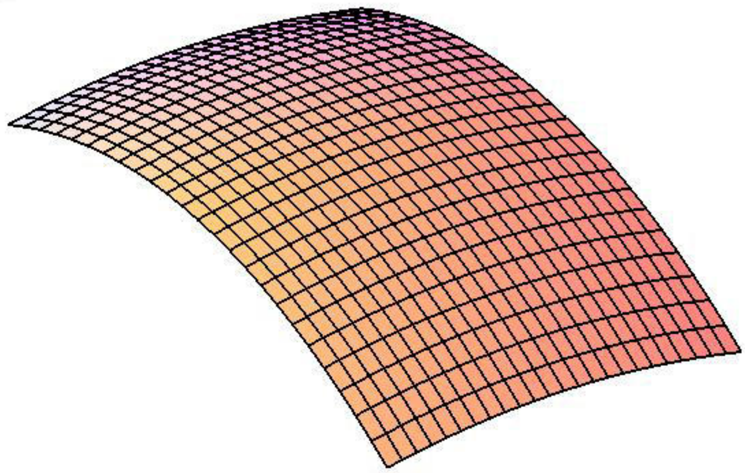
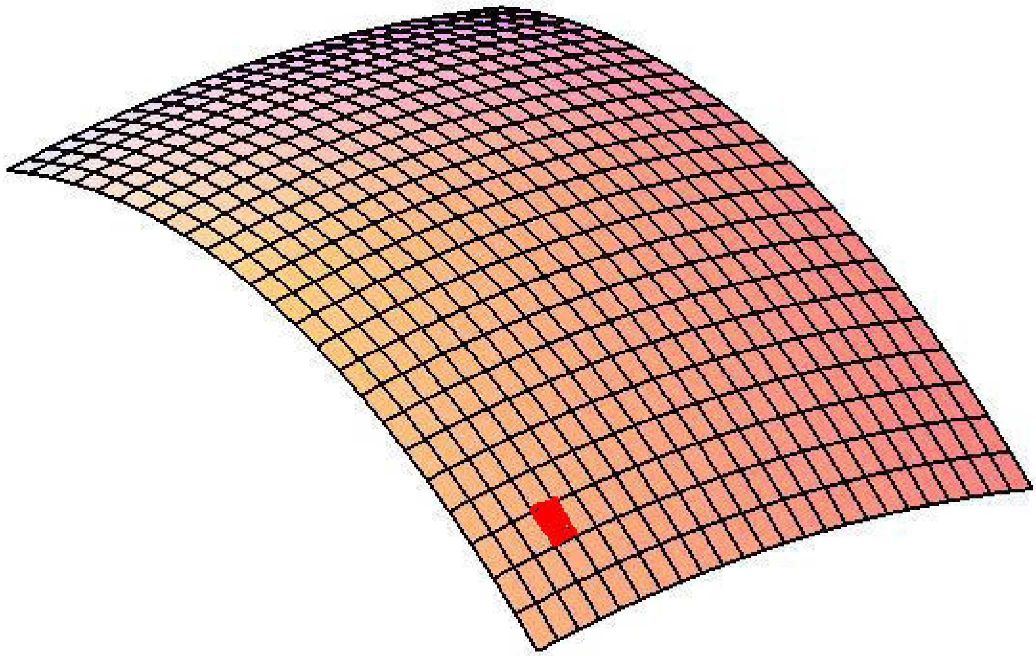
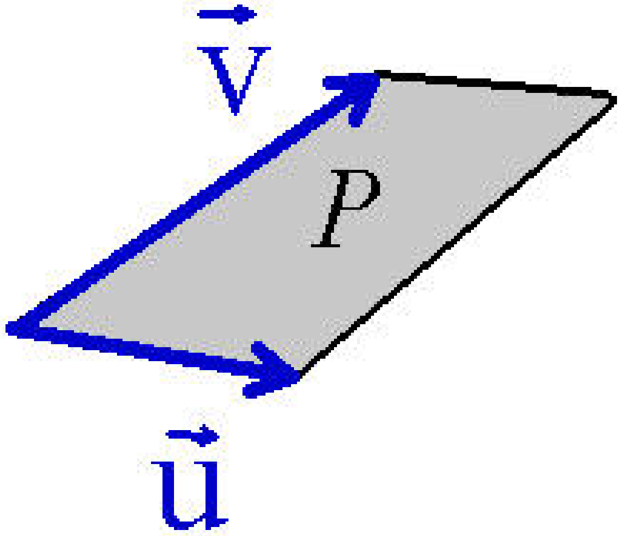
Surface Area

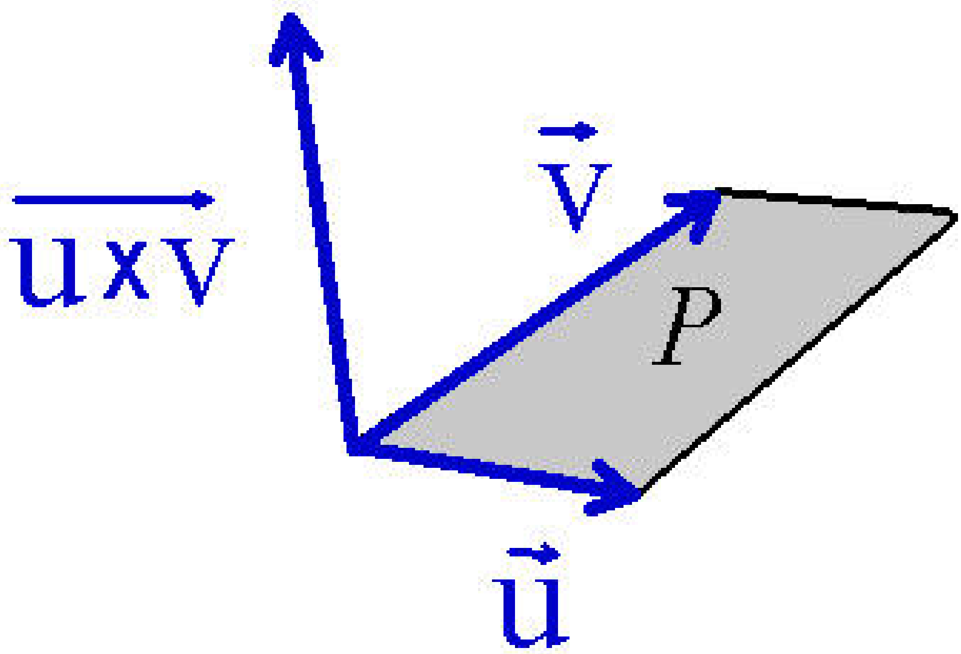
$$z = f(x, y)$$



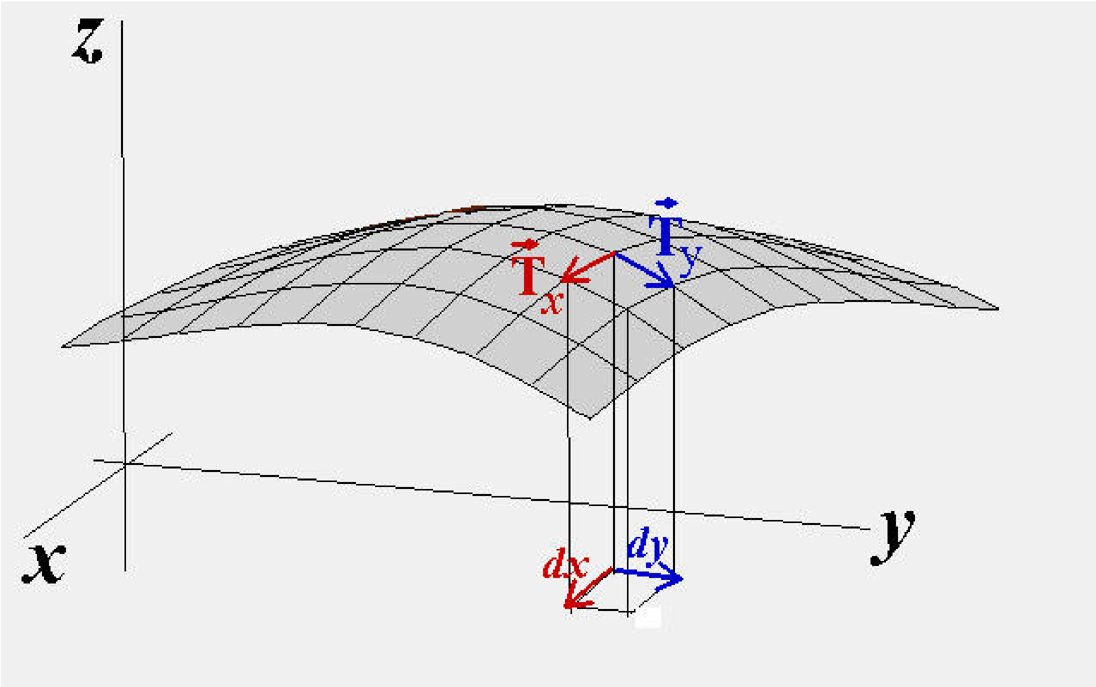




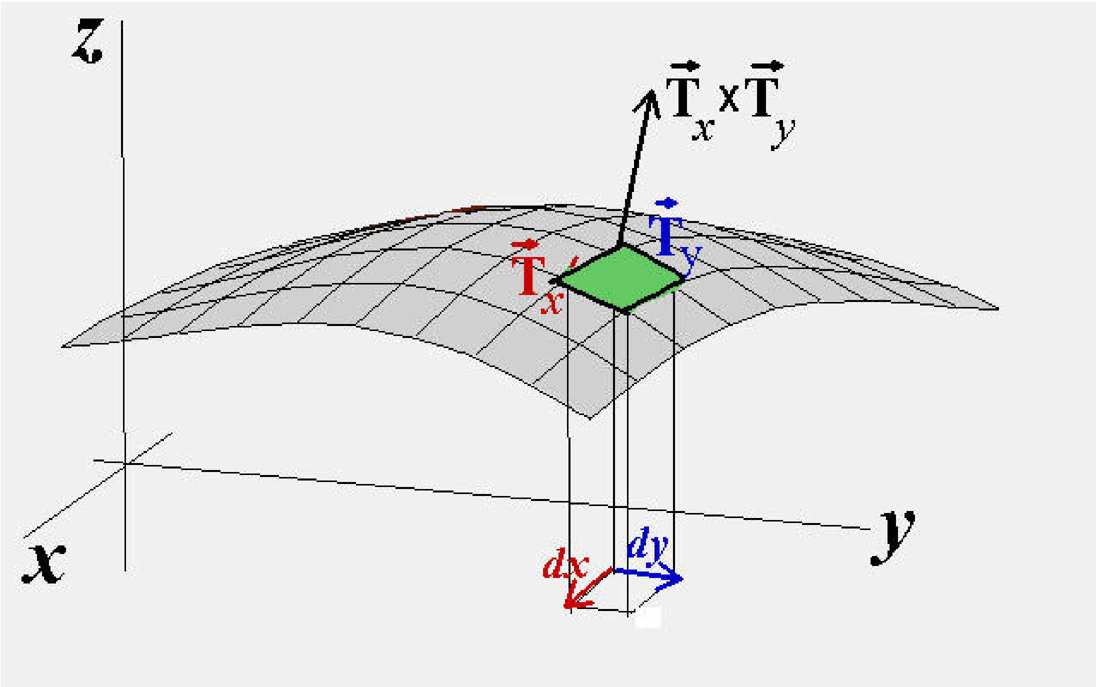
$\vec{v}$
P
$\vec{u}$



u x v
v
P
u



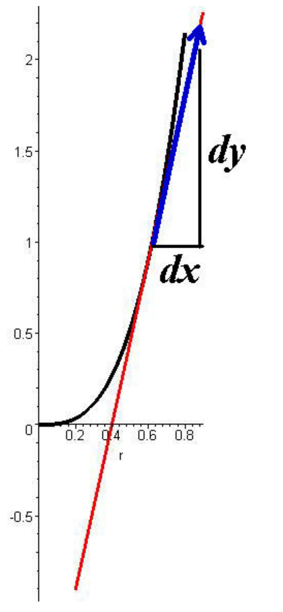
z
x
y
T
x
T
y
dx
dy



z
$\vec{T}_x \times \vec{T}_y$
$\vec{T}_y$
$\vec{T}_x$
x
y
dx
dy

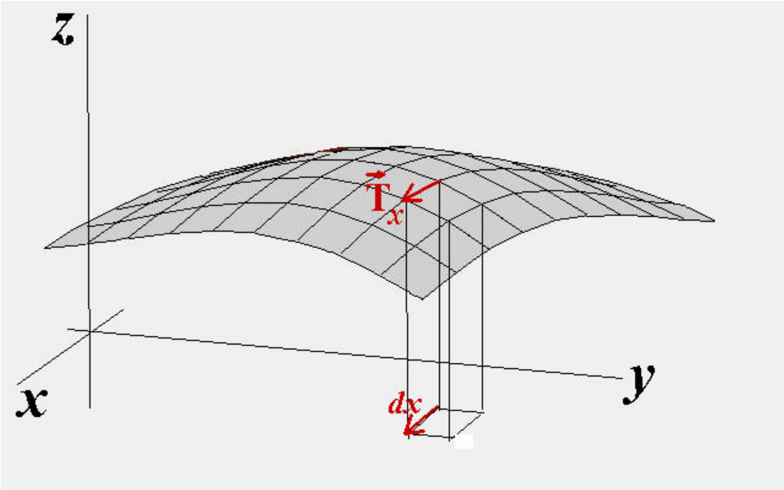
Let $\vec{\mathbf{r}} = \langle x,\ y\rangle$ be the position vector

$$\vec{\mathbf{T}} = \langle dx,\ dy\rangle = \left\langle 1,\ \frac{dy}{dx}\right\rangle dx = \frac{d\vec{\mathbf{r}}}{dx}\,dx$$



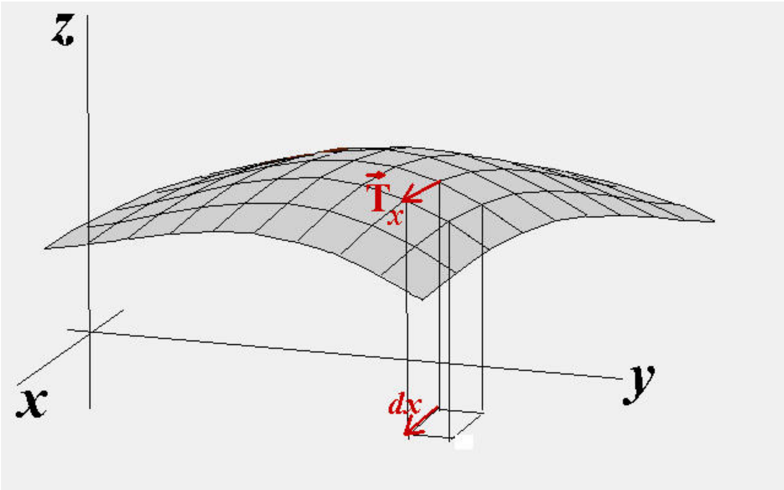
Let $\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle$ be the position vector

$$\vec{\mathbf{T}}_x = \frac{\partial \vec{\mathbf{r}}}{\partial x}\, dx = \left\langle 1,\ 0,\ \frac{\partial z}{\partial x} \right\rangle dx$$



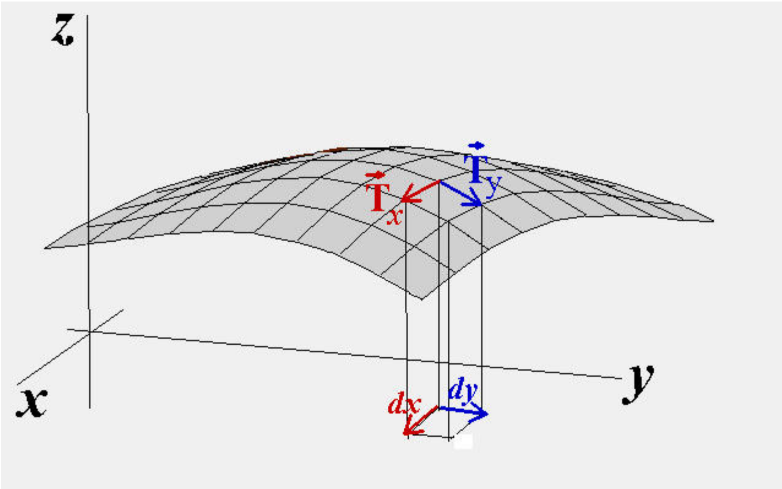
Let $\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle$ be the position vector

$$\vec{\mathbf{T}}_x = \frac{\partial \vec{\mathbf{r}}}{\partial x}\, dx = \langle 1,\ 0,\ z_x \rangle\, dx$$

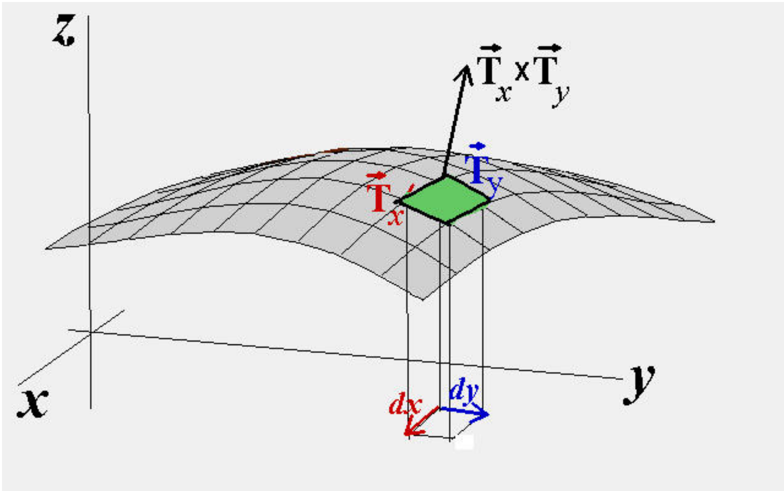


Let $\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle$ be the position vector

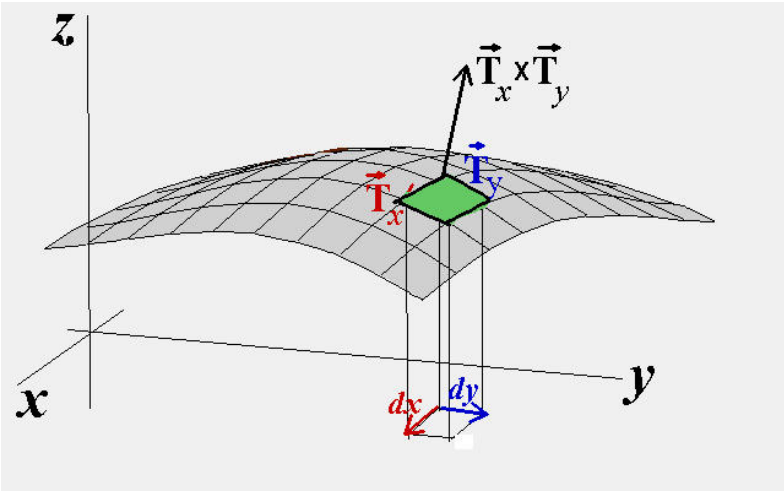
$$\vec{\mathbf{T}}_y = \frac{\partial \vec{\mathbf{r}}}{\partial y}\, dy = \langle 0,\ 1,\ z_y \rangle\, dy$$

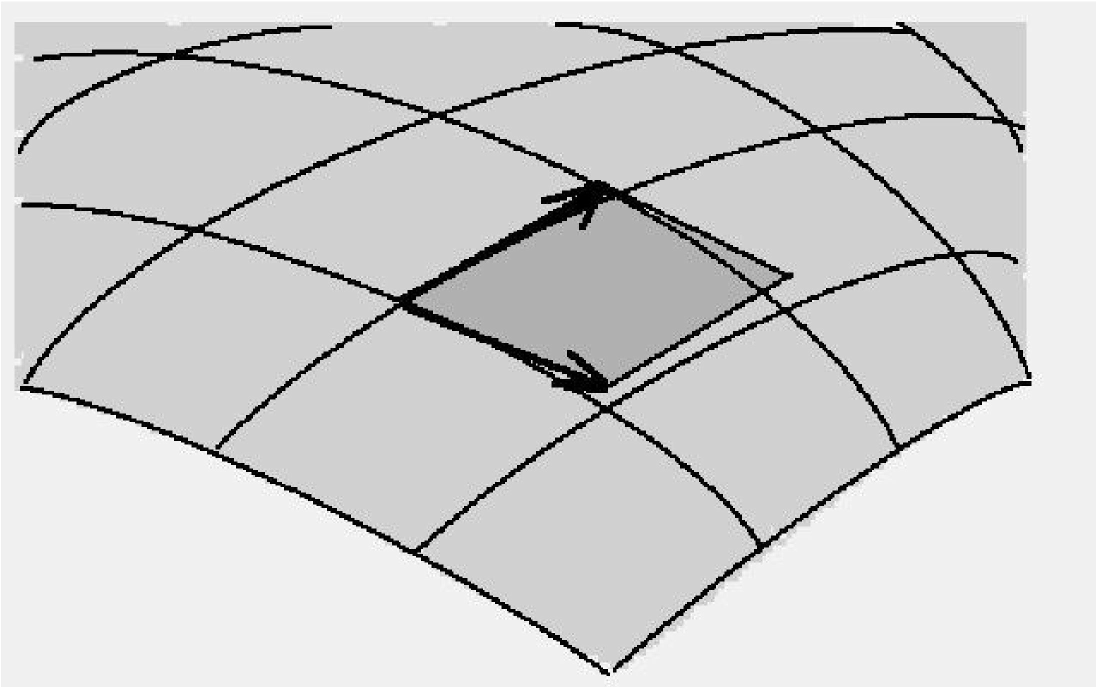


$$\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y$$



$$dS = |\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y|$$



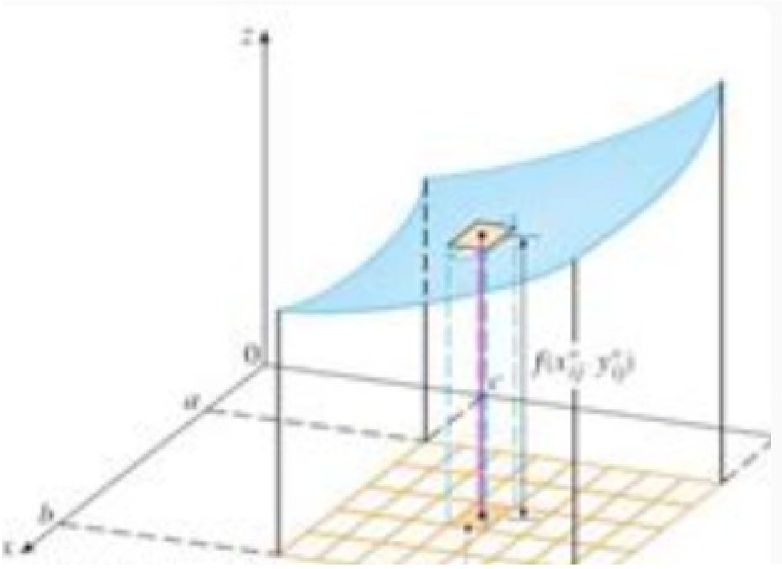


$$
\begin{aligned}
\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y &= \langle 1,\ 0,\ z_x \rangle \times \langle 0,\ 1,\ z_y \rangle \; dx\,dy \\
&= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} dx\,dy \\
&= \langle -z_x,\ -z_y,\ 1 \rangle \, dx\,dy
\end{aligned}
$$

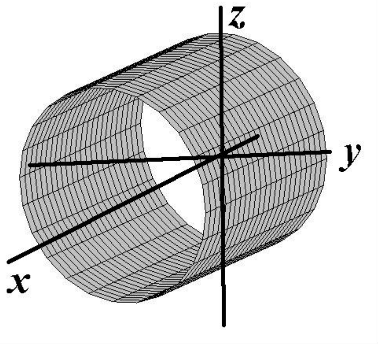
$$\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y = \langle 1,\ 0,\ z_x \rangle \times \langle 0,\ 1,\ z_y \rangle\ dx\,dy$$
$$= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} dx\,dy$$
$$= \langle -z_x,\ -z_y,\ 1 \rangle\, dx\,dy$$

$$dS = |\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_y| = \sqrt{z_x^2 + z_y^2 + 1}\, dx\,dy$$

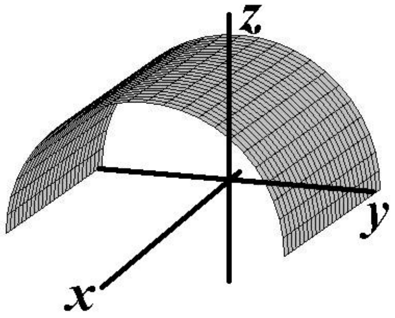
$$\mathrm{A}(S) = \iint_{\mathcal{D}} \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy$$



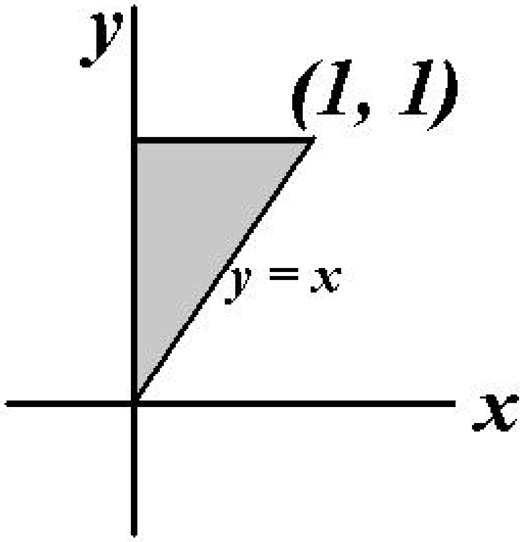
$$y^2 + z^2 = 1$$



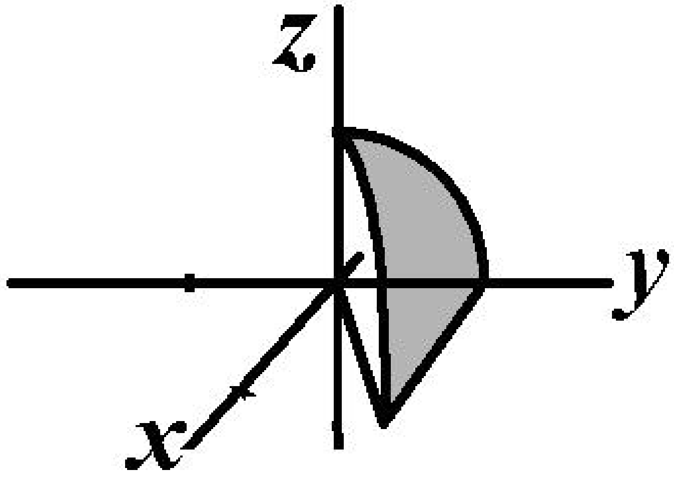
$$z = \sqrt{1 - y^2}$$



Let $\mathcal{D}$ be the triangular region in the xy plane



Let S be the portion of the cylinder $z = \sqrt{1 - y^2}$ that is directly above $\mathcal{D}$

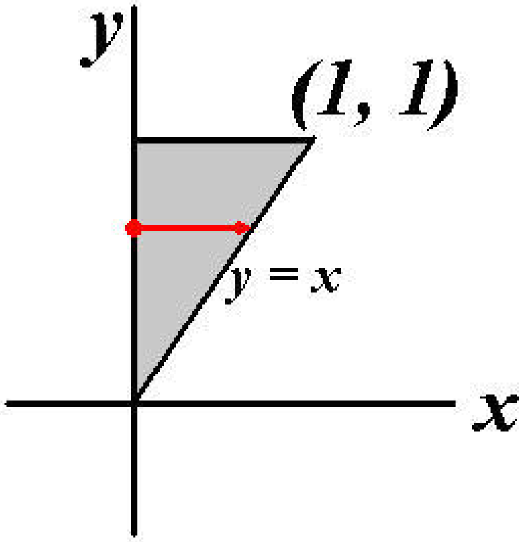


$$\mathrm{A}(S) = \iint_{\mathcal{D}} \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy$$

If $z = \sqrt{1 - y^2}$ then: $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1-y^2}}$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{0 + \frac{y^2}{1 - y^2} + 1} = \frac{1}{\sqrt{1 - y^2}}$$

$$\mathrm{A}(S) = \iint_{\mathcal{D}} \sqrt{z_x^2 + z_y^2 + 1}\, dA$$
$$= \int_0^1 \int_0^y \frac{1}{\sqrt{1 - y^2}}\, dx\, dy$$

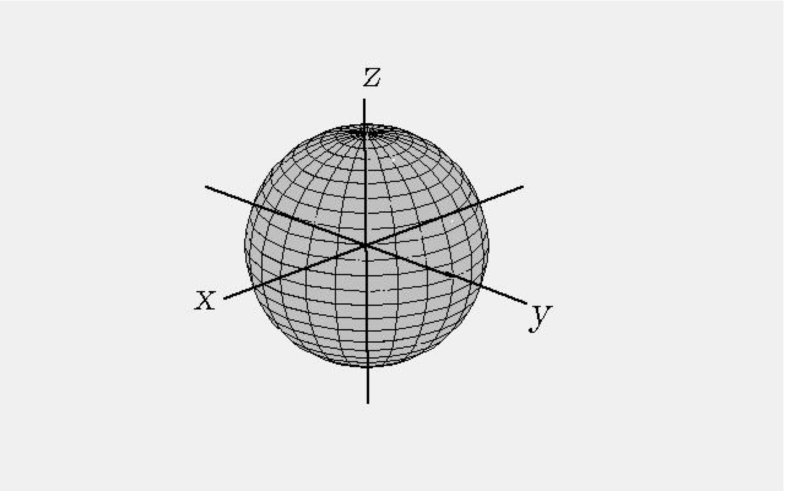


$$
\begin{aligned}
\mathrm{A}(S) &= \iint_{\mathcal{D}} \sqrt{z_x^2 + z_y^2 + 1}\, dA \\
&= \int_0^1 \int_0^y \frac{1}{\sqrt{1-y^2}}\, dx\, dy \\
&= \int_0^1 \frac{y}{\sqrt{1-y^2}}\, dy \\
&= 1
\end{aligned}
$$

Example:

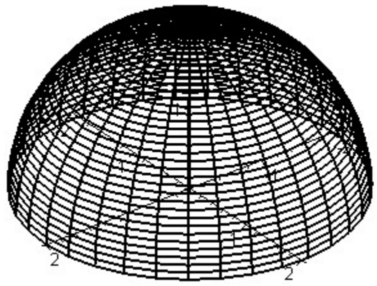
Calculate the surface area of a sphere of radius a

$$x^2 + y^2 + z^2 = a^2$$



We can always double the surface area of a hemisphere

$$z = \sqrt{a^2 - x^2 - y^2}$$



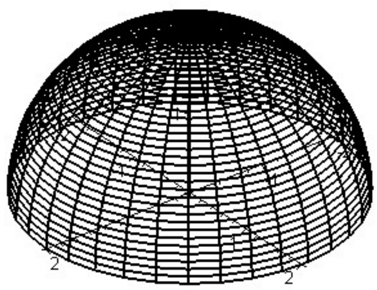
$$z = \left(a^2 - x^2 - y^2\right)^{1/2}$$

$$z_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \qquad z_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\begin{aligned} z_x^2 + z_y^2 + 1 &= \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1 \\ &= \frac{a^2}{a^2 - x^2 - y^2} \end{aligned}$$

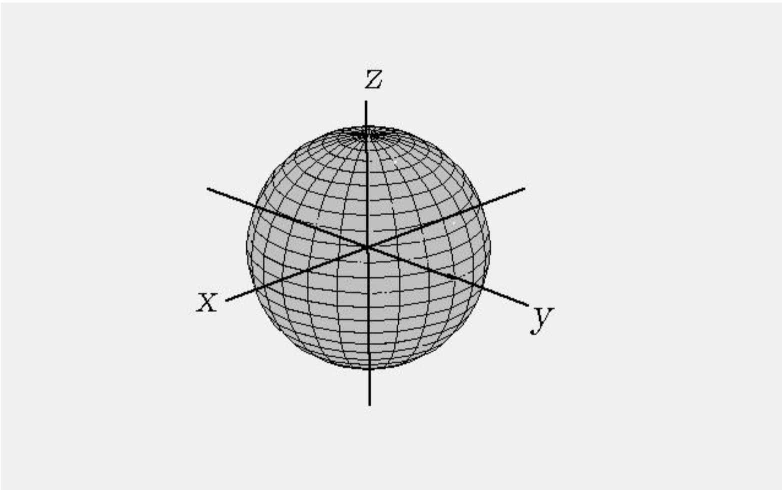
$$dS = \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy$$

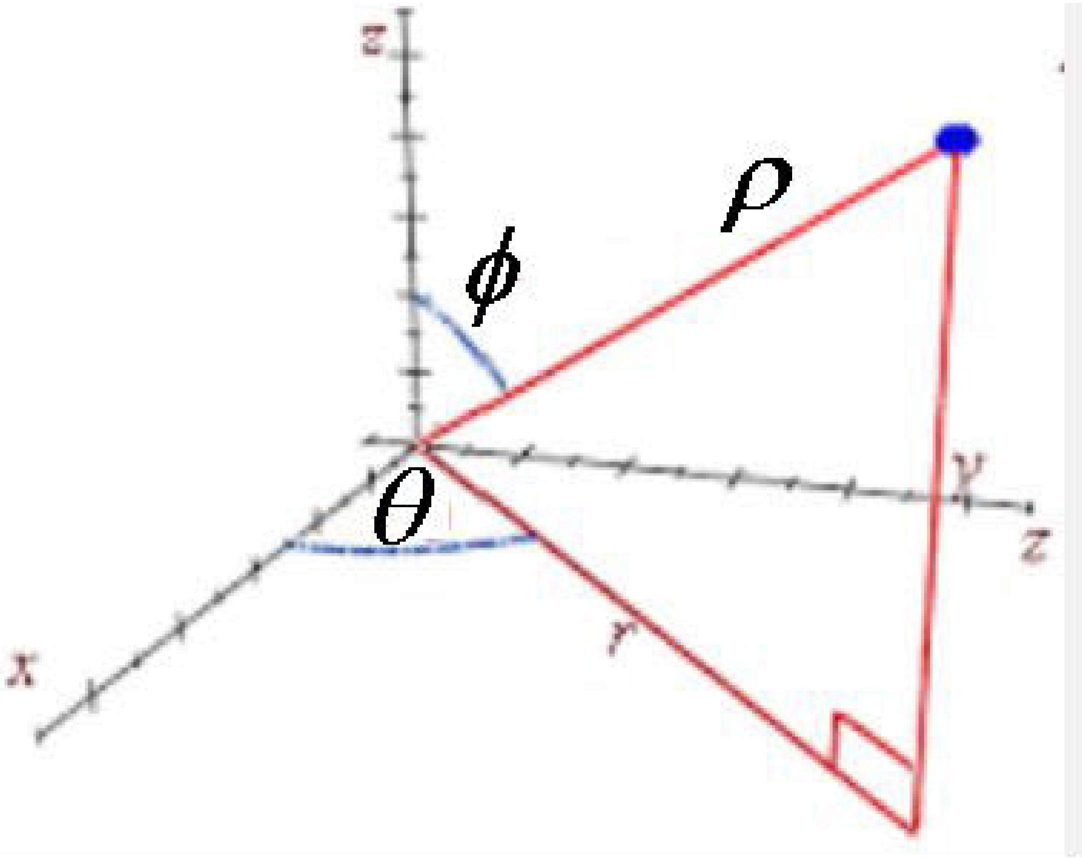
$$\text{Area}(S) = \iint_{\mathcal{D}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy$$



$$\begin{aligned}
\text{Area}(S) &= \iint_{\mathcal{D}} \frac{a}{\sqrt{a^2 - (x^2 + y^2)}}\, dA \\
&= \int_0^a \int_0^{2\pi} \frac{a}{\sqrt{a^2 - r^2}}\, r\, d\theta\, dr \\
&= \int_0^a \frac{2\pi a r}{\sqrt{a^2 - r^2}}\, dr \\
&= 2\pi a^2
\end{aligned}$$

$$\text{Area(Sphere)} = 2 \cdot \text{Area(hemisphere)} = 4\pi a^2$$




z
ϕ
ρ
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y
z
x
r