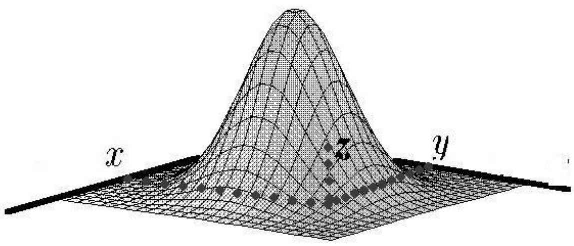
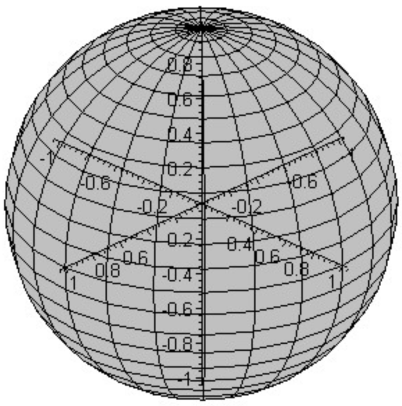
Surface Area - Parametric

$$\text{Area}(S) = \iint_{\mathcal{D}} \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy$$



$$x^2 + y^2 + z^2 = 1$$

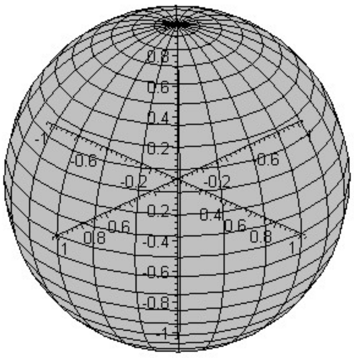
This can't be written in the form $z = f(x, y)$

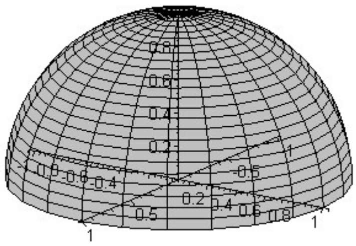


$$x^2 + y^2 + z^2 = 1$$

This can't be written in the form $z = f(x, y)$

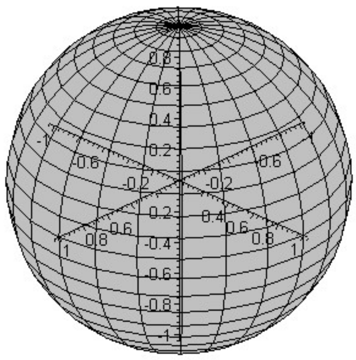
$$z = \sqrt{1 - x^2 - y^2}$$

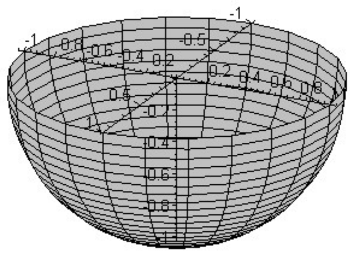


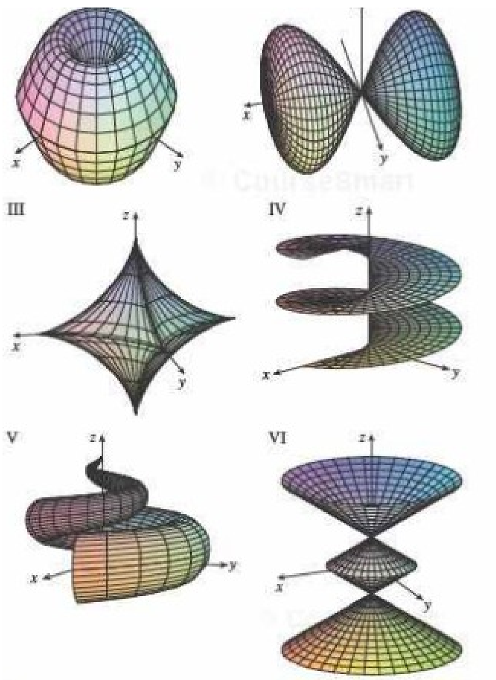


This can't be written in the form $z = f(x, y)$

$$z = -\sqrt{1 - x^2 - y^2}$$

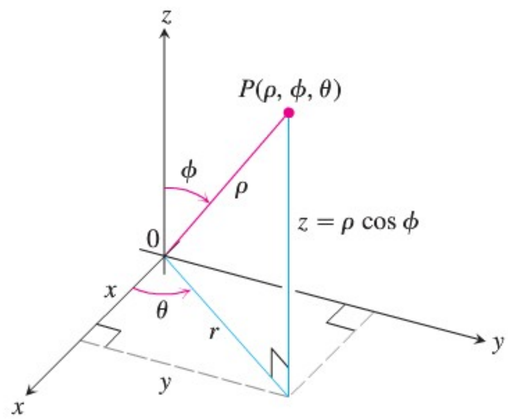




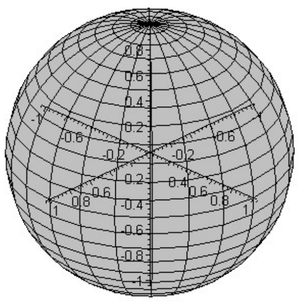


III
IV
V
VI

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$



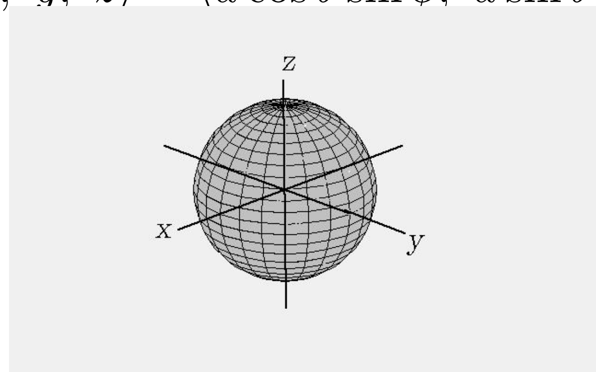
$$x = a\cos\theta\sin\phi$$
$$y = a\sin\theta\sin\phi$$
$$z = a\cos\phi$$



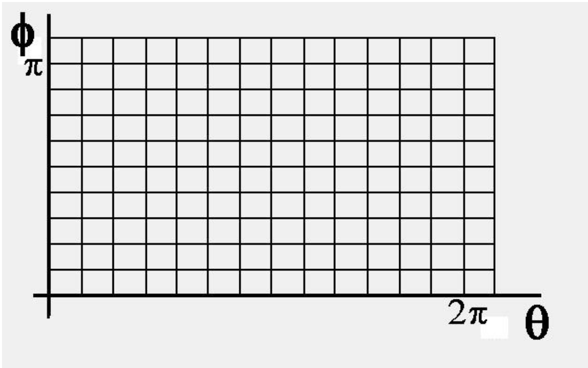
$$x = a\cos\theta\sin\phi \qquad y = a\sin\theta\sin\phi \qquad z = a\cos\phi$$

In vector form,

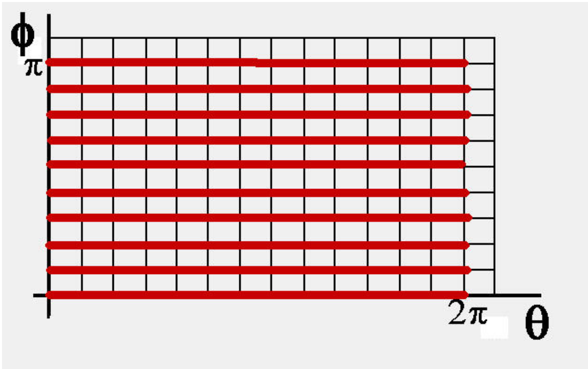
$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi\rangle$$



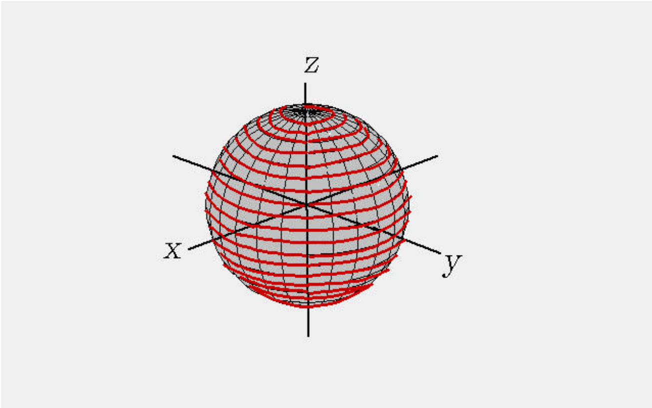
$$0 \leq \theta \leq 2\pi \qquad\qquad 0 \leq \phi \leq \pi$$



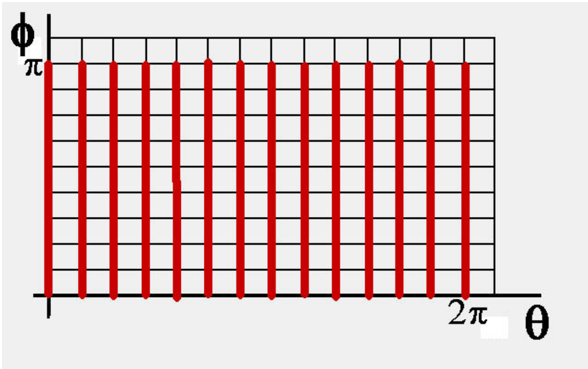
Hold ϕ constant and vary θ



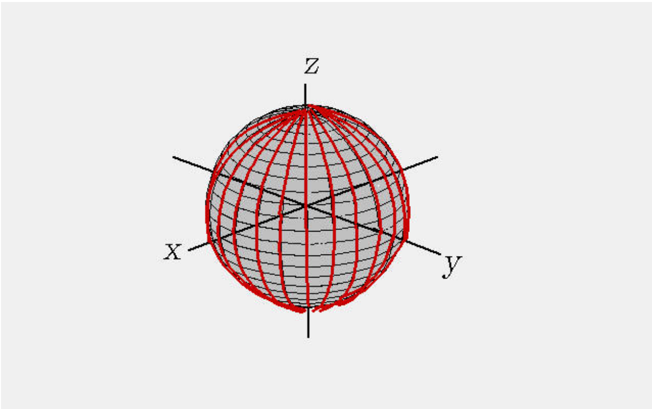
Hold ϕ constant and vary θ



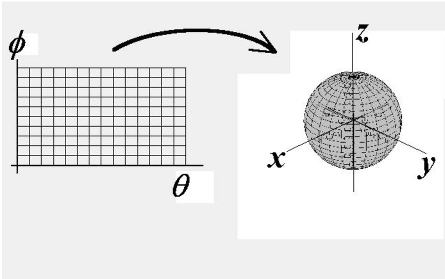
Hold θ constant and vary ϕ



Hold θ constant and vary ϕ



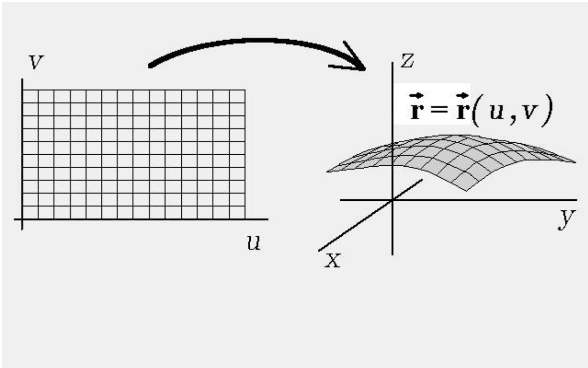
$$\vec{\mathbf{r}} = \langle\, x(\phi,\, \theta),\;\, y(\phi,\, \theta), z(\phi,\, \theta)\,\rangle$$



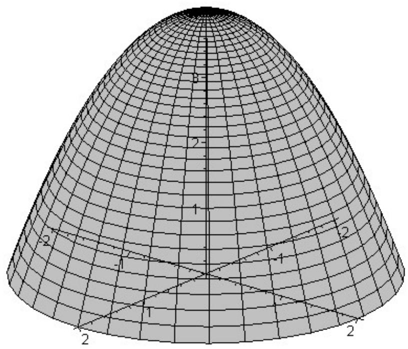
More generally, surfaces can be described by a vector equation of the form

$$\vec{\mathbf{r}} = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$$

The variables u and v are called the parameters



$$z = 4 - x^2 - y^2$$



Polar Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

Polar Coordinates

$$x = u\cos v \qquad y = u\sin v$$

$$z = 4 - x^2 - y^2 = 4 - \left(x^2 + y^2\right)$$

$$\text{Let} \quad x = u\cos v \qquad y = u\sin v$$

$$z = 4 - \left((u\cos v)^2 + (u\sin v)^2\right) = 4 - u^2$$

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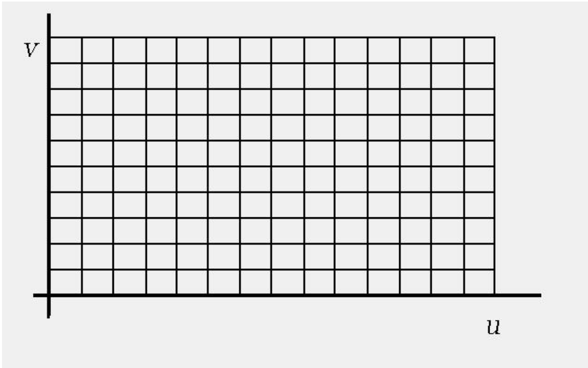
$$\text{Let} \quad x = u\cos v \qquad y = u\sin v$$

$$z = 4 - \left((u\cos v)^2 + (u\sin v)^2\right) = 4 - u^2$$

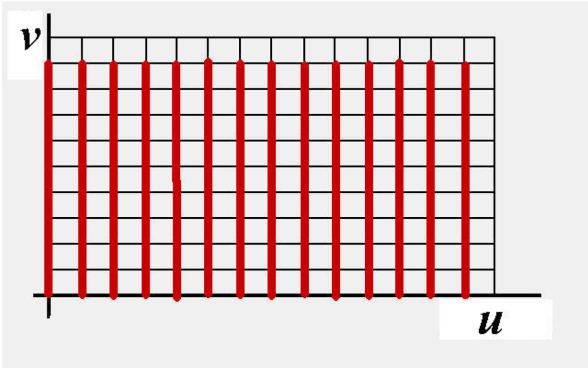
$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle u\cos v,\ u\sin v,\ 4 - u^2\rangle$$

$$\text{where } 0 \le v \le 2\pi \text{ and } 0 \le u \le 2$$

$0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$



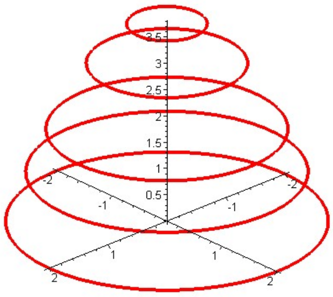
$0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$



$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle u\cos v,\ u\sin v,\ 4 - u^2 \rangle$$

where $0 \le v \le 2\pi$ and $0 \le u \le 2$

Vary v for different values of u

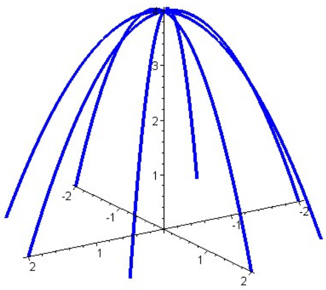


$0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle u\cos v,\ u\sin v,\ 4 - u^2 \rangle$$

where $0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$

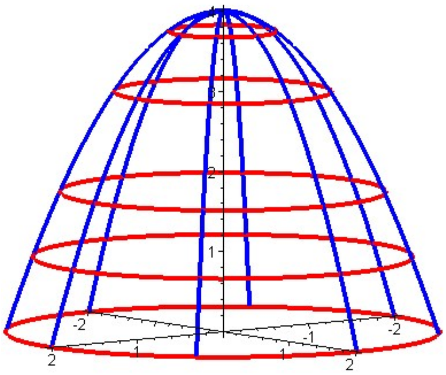
Vary u for different values of v



$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle u\cos v,\ u\sin v,\ 4 - u^2 \rangle$$

where $0 \leq v \leq 2\pi$ and $0 \leq u \leq 2$

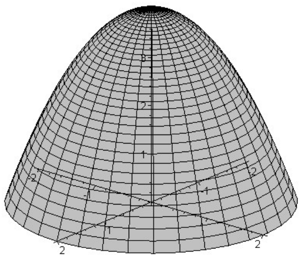
Combine the grid lines together



$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle u\cos v,\ u\sin v,\ 4-u^2\rangle$$

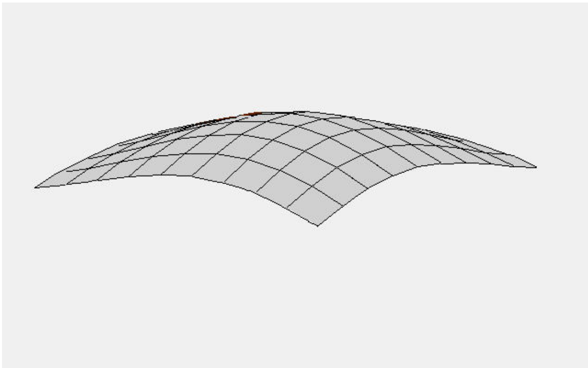
where $0 \le v \le 2\pi$ and $0 \le u \le 2$

We have induced a grid on the parabolic surface

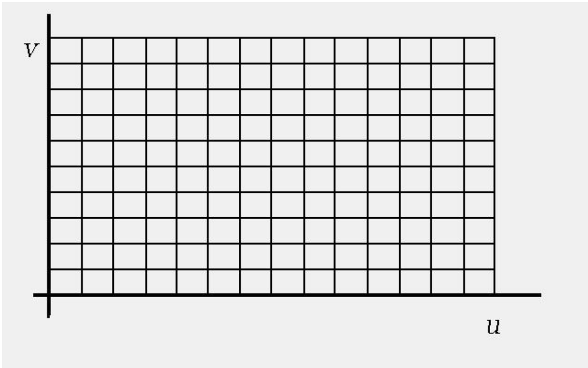


Area of the Surface Element

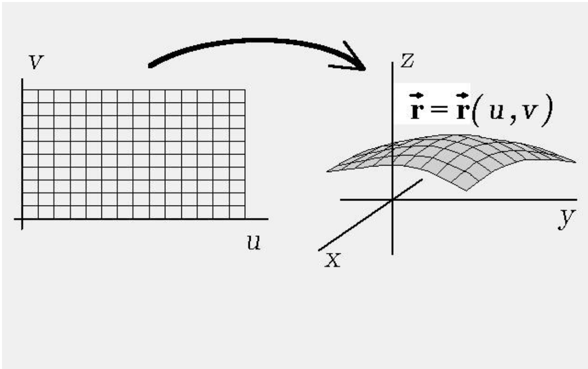
The equation $\vec{\mathbf{r}}(u,v) = \langle x,\, y,\, z\rangle$ represents a transformation from a uv-plane to an xyz-axis. The graph of all the points $\vec{\mathbf{r}}(u,v)$ is a surface.



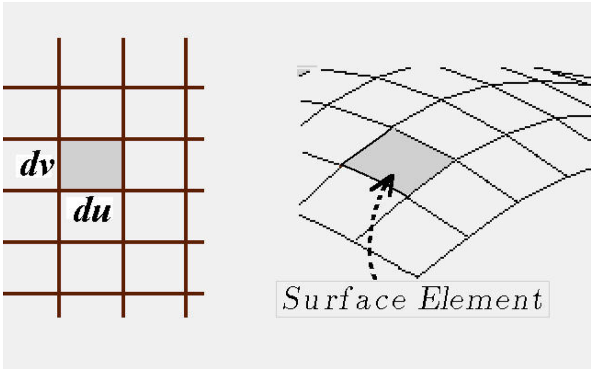
Suppose we subdivide a region of the uv-axis into a rectangular grid.



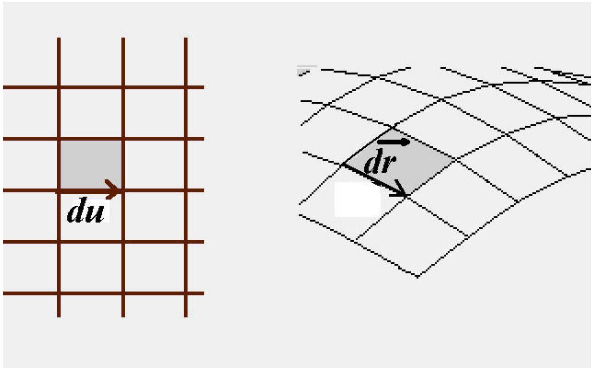
This will automatically subdivide the surface $\vec{\mathbf{r}}(u, v)$ into a grid.



Consider one rectangle on the uv grid.

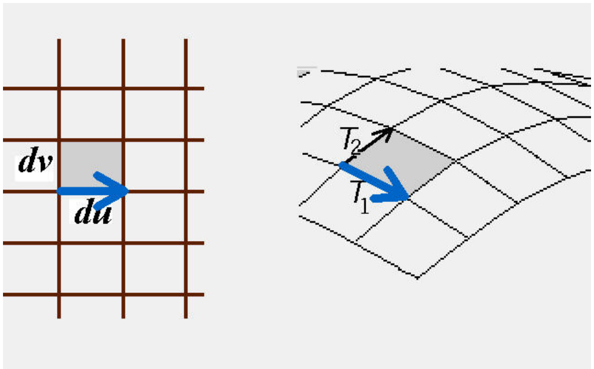


Let $\vec{\mathbf{dr}}$ be the vector denoting the change in position along this curve.

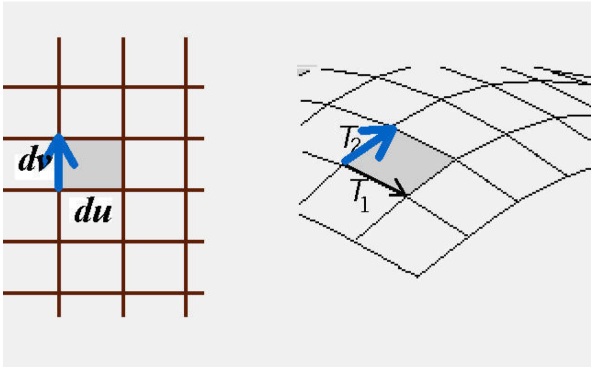


$$\text{Let } \vec{\mathbf{T}}_1 = \mathbf{d}\vec{\mathbf{r}} = \frac{\partial \vec{\mathbf{r}}}{\partial u}\, du$$

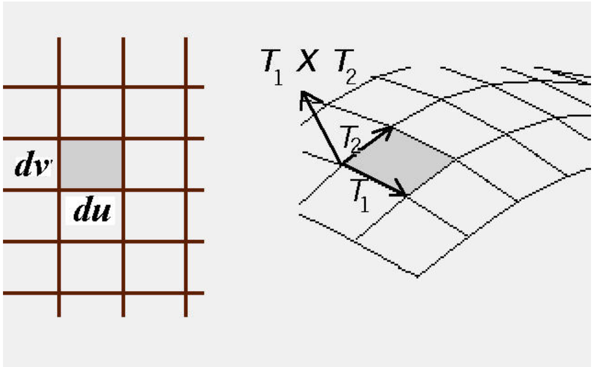
$$\vec{\mathbf{T}}_1 = \frac{\partial \vec{\mathbf{r}}}{\partial u} du = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle du$$



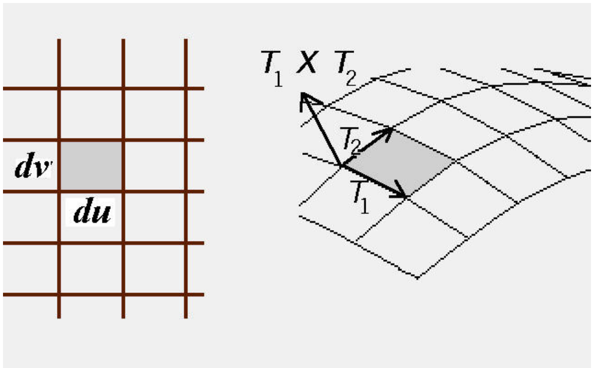
$$\vec{\mathbf{T}}_2 = \frac{\partial \vec{\mathbf{r}}}{\partial v} dv = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle dv$$



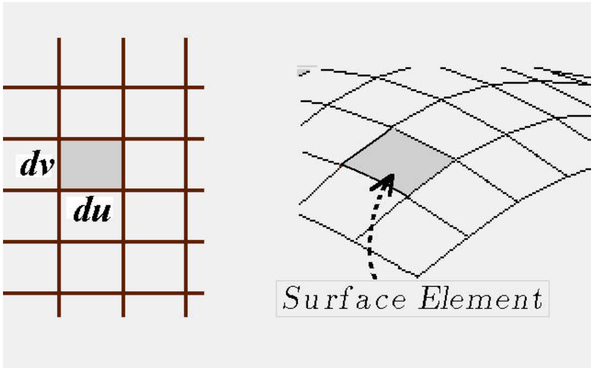
$$\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2 = \left(\frac{\partial \vec{\mathbf{r}}}{\partial u} du\right) \times \left(\frac{\partial \vec{\mathbf{r}}}{\partial v} dv\right)$$



$$\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2 = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du\, dv$$

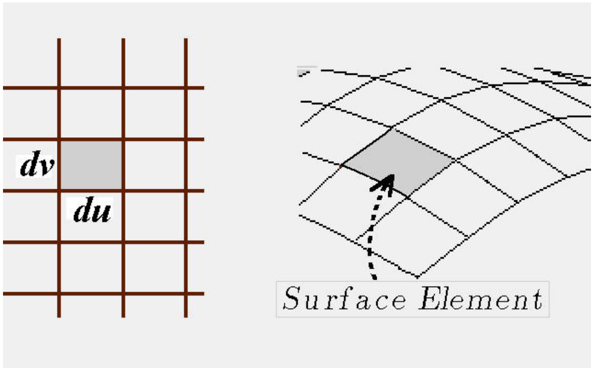


The length of this cross product is equal to the area of one surface element

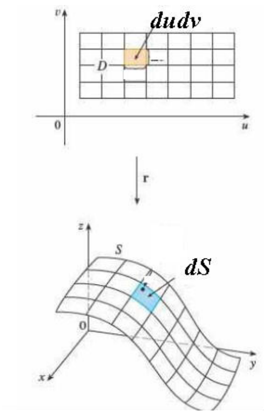


The length of this cross product is denoted by dS

$$dS = \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right| du\, dv$$

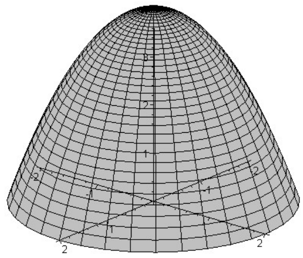


$$\text{Total Surface Area of } S = \iint_{\mathcal{D}} \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right| du\, dv$$



$$\vec{\mathbf{r}} = \left\langle u\cos v,\ u\sin v,\ 4 - u^2 \right\rangle$$

where $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$



$$\vec{\mathbf{r}} = \left\langle u\cos v,\ u\sin v,\ 4 - u^2 \right\rangle$$

$$\vec{\mathbf{r}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u} = \langle \cos v,\ \sin v,\ -2u \rangle$$

$$\vec{\mathbf{r}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v} = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$\vec{\mathbf{r}} = \left\langle u\cos v,\ \ u\sin v,\ \ 4 - u^2 \right\rangle$$

$$\vec{\mathbf{r}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u} = \langle \cos v,\ \sin v,\ -2u \rangle$$

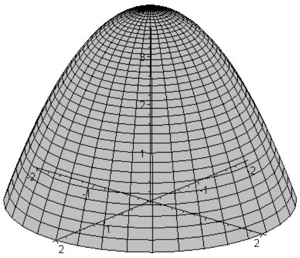
$$\vec{\mathbf{r}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v} = \langle -u\sin v,\ u\cos v,\ \ 0 \rangle$$

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \left\langle 2u^2\cos v,\ \ 2u^2\sin v,\ \ u \right\rangle$$

$$|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| = \sqrt{4u^4 + u^2} = u\sqrt{4u^2 + 1}$$

Let S be the portion of the paraboloid for $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$ Find the surface area

$$\mathrm{A}(S) = \iint_{\mathcal{D}} |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dv \, du$$



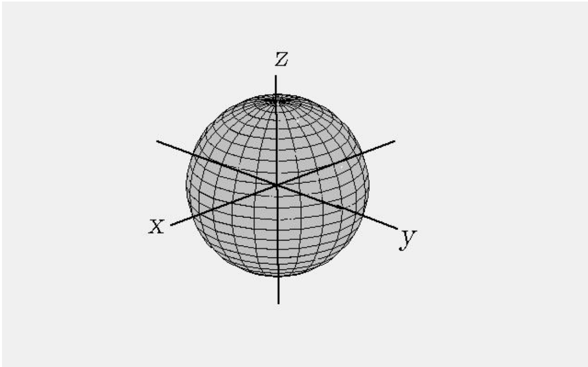
$$
\begin{aligned}
\mathrm{A}(S) &= \iint_{\mathcal{D}} |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dv \, du \\
&= \int_0^2 \int_0^{2\pi} u\sqrt{4u^2+1} \, dv \, du \\
&= \int_0^2 2\pi u\sqrt{4u^2+1} \, du \\
&= \frac{\pi}{6}(17\sqrt{17}-1)
\end{aligned}
$$

Find the surface area of a sphere of radius a

$$x = a\cos\theta\sin\phi \qquad y = a\sin\theta\sin\phi \qquad z = a\cos\phi$$

In vector form,

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi\rangle$$



$$\mathrm{A}(S) = \iint_{\mathcal{D}} |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dv \, du$$

$$A(S) = \int_0^{\pi} \int_0^{2\pi} |\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta| \, d\theta \, d\phi$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi\rangle$$

$$\vec{\mathbf{r}}_\phi = \langle a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -a\sin\phi\rangle$$
$$\vec{\mathbf{r}}_\theta = \langle -a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0\rangle$$

Calculate $\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta$

$$\vec{\mathbf{r}}_\phi = \langle a\cos\theta\cos\phi,\ \ a\sin\theta\cos\phi,\ \ -a\sin\phi\rangle$$
$$\vec{\mathbf{r}}_\theta = \langle -a\sin\theta\sin\phi,\ \ a\cos\theta\sin\phi,\ \ 0\rangle$$

$$\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \end{vmatrix}$$

$$\vec{\mathbf{r}}_\phi = \langle a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -a\sin\phi\rangle$$
$$\vec{\mathbf{r}}_\theta = \langle -a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0\rangle$$

$$\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \end{vmatrix}$$
$$= a^2\sin\phi\,\langle\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$$

$$|\langle \cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi \rangle| = 1$$

$$\begin{aligned} |\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta| &= a^2 \sin\phi |\langle \cos\theta \sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle| \\ &= a^2 \sin\phi \end{aligned}$$

$$dS = |\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta|\, d\theta\, d\phi = a^2 \sin\phi\, d\theta\, d\phi$$

$$\begin{aligned}
\mathrm{A}(S) &= \int_0^\pi \int_0^{2\pi} a^2 \sin\phi\, d\theta\, d\phi \\
&= 2\pi a^2 \int_0^\pi \sin\phi\, d\phi \\
&= 4\pi a^2
\end{aligned}$$