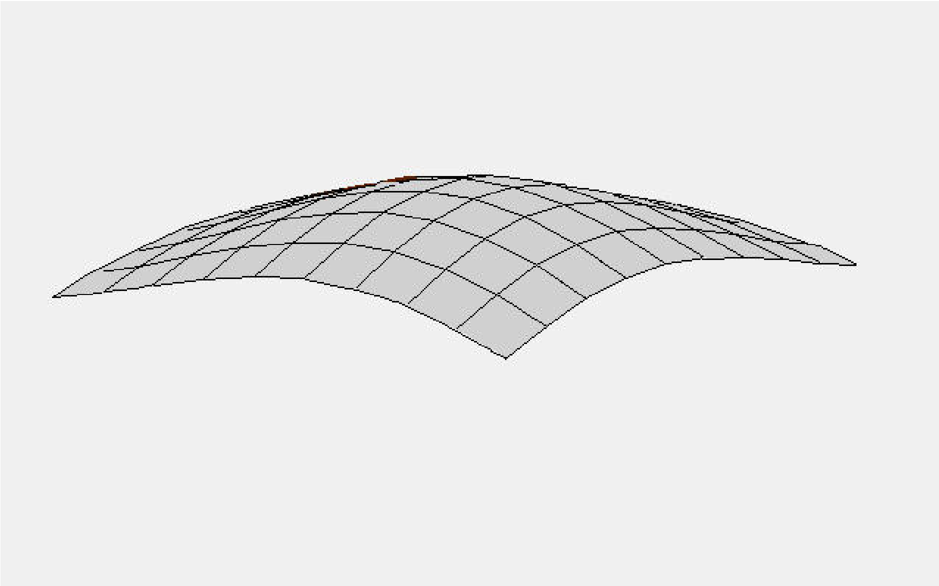
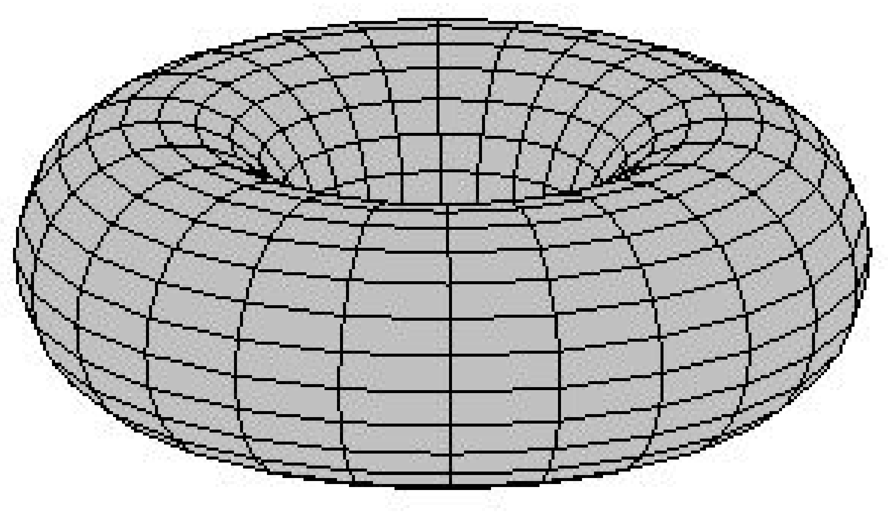
Find the area of a surface S

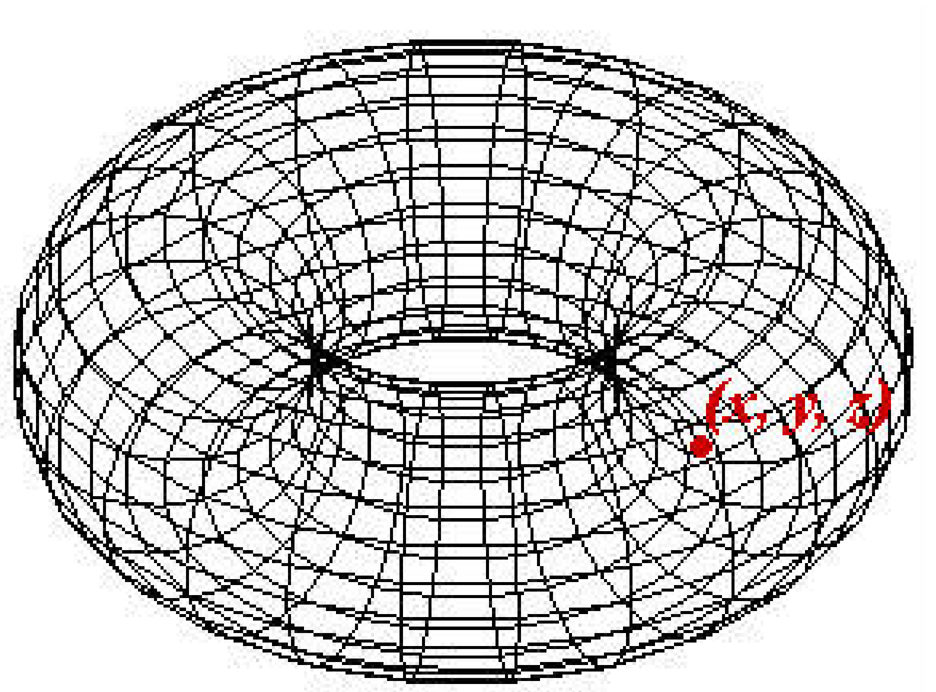
$$A(S) = \iint_{\mathcal{D}} \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right| du\, dv$$



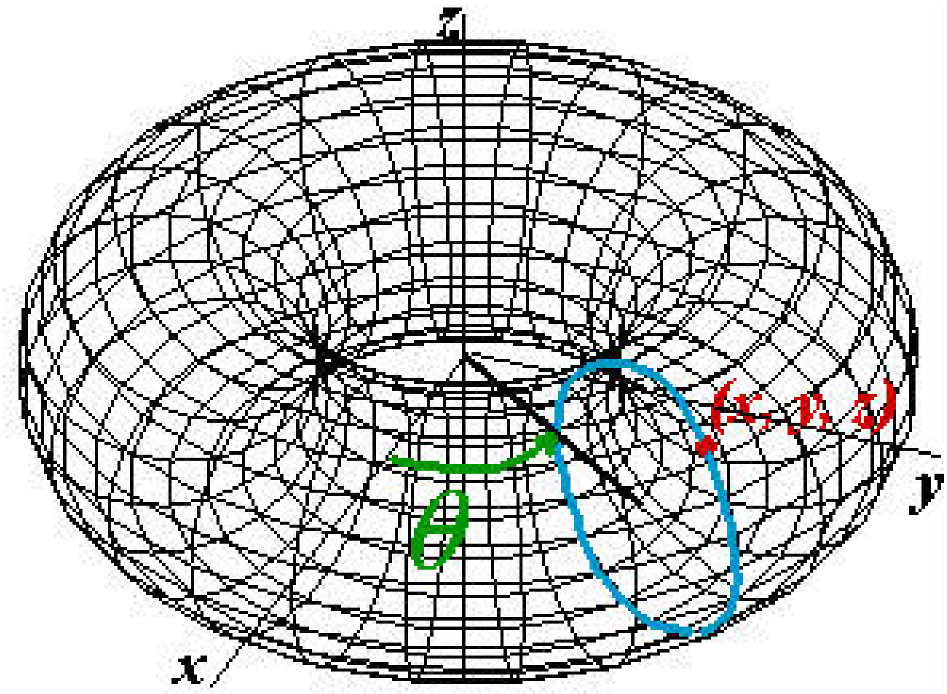
Find the surface area of a torus.
Assume the radius of the “hole” in the torus is 1 and the radius of the circular cross-section is also 1.



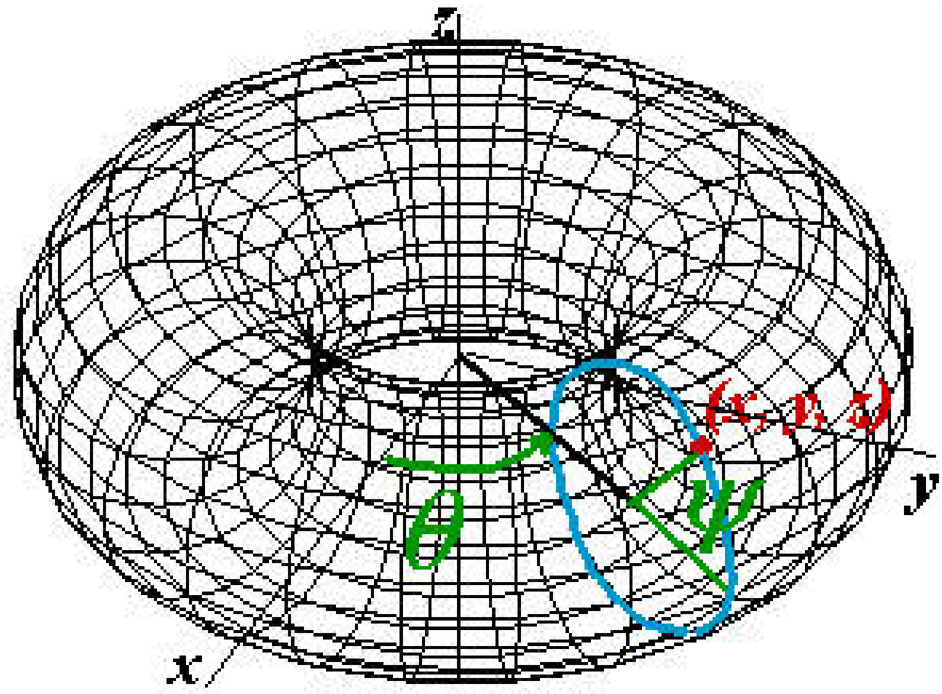
Describe each position vector $\vec{\mathbf{r}}$ in terms of two parameters



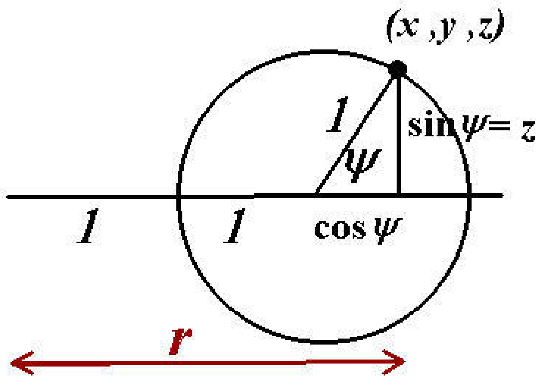
Take a cross-section of the torus. Let θ be the angle this cross-section makes with the positive x-axis.



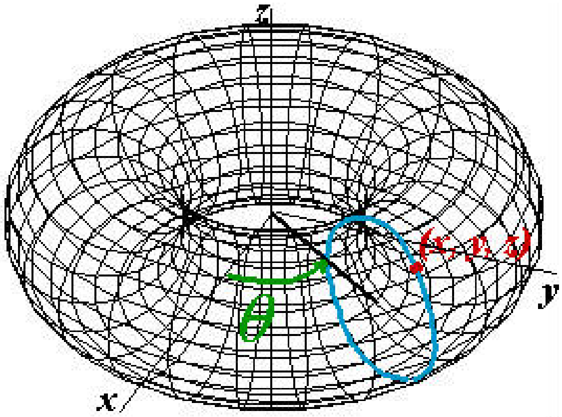
Let ψ be the angle the radius vector of the cross-section makes with the xy-plane.



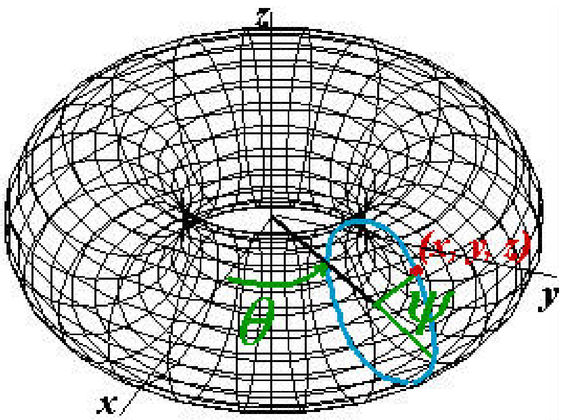
$$z = \sin\psi$$
$$r = 2 + \cos\psi$$



$$x = r\cos\theta = (2+\cos\psi)\cos\theta$$
$$y = r\sin\theta = (2+\cos\psi)\sin\theta$$
$$z = \sin\psi$$

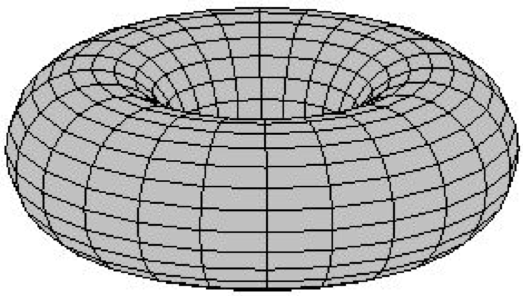


$$\vec{\mathbf{r}} = \langle (2+\cos\psi)\cos\theta,\ (2+\cos\psi)\sin\theta,\ \sin\psi \rangle$$



$$\vec{\mathbf{r}} = \langle (2+\cos\psi)\cos\theta,\ (2+\cos\psi)\sin\theta,\ \sin\psi \rangle$$

$$\text{Area(Torus)} = \int_0^{2\pi}\int_0^{2\pi} \left| \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \psi} \right| \, d\theta \, d\psi$$



$$\vec{\mathbf{r}} = \langle (2+\cos\psi)\cos\theta,\ (2+\cos\psi)\sin\theta,\ \sin\psi \rangle$$

$$\text{We need to calculate } \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \psi}$$

$$\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -(2+\cos\psi)\sin\theta & (2+\cos\psi)\cos\theta & 0 \\ -\sin\psi\cos\theta & -\sin\psi\sin\theta & \cos\psi \end{vmatrix}$$

$$\vec{\mathbf{r}} = \langle (2+\cos\psi)\cos\theta,\ (2+\cos\psi)\sin\theta,\ \sin\psi \rangle$$

We need to calculate $\dfrac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \dfrac{\partial \vec{\mathbf{r}}}{\partial \psi}$

$$\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -(2+\cos\psi)\sin\theta & (2+\cos\psi)\cos\theta & 0 \\ -\sin\psi\cos\theta & -\sin\psi\sin\theta & \cos\psi \end{vmatrix}$$

$$= (2+\cos\psi)\langle \cos\theta\cos\psi,\ \sin\theta\cos\psi,\ \sin\psi \rangle$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \psi} = (2 + \cos\psi)\langle \cos\theta\cos\psi,\ \sin\theta\cos\psi,\ \sin\psi \rangle$$

$\left| \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \psi} \right|$ will be:

$$(2 + \cos\psi)\sqrt{\cos^2\theta\cos^2\psi + \sin^2\theta\cos^2\psi + \sin^2\psi}$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \psi} = (2 + \cos\psi)\langle \cos\theta \cos\psi,\ \sin\theta\cos\psi,\ \sin\psi \rangle$$

$\left| \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \psi} \right|$ will be:

$$(2 + \cos\psi)\sqrt{\cos^2\theta\cos^2\psi + \sin^2\theta\cos^2\psi + \sin^2\psi}$$

$$= (2 + \cos\psi)\sqrt{\left(\cos^2\theta + \sin^2\theta\right)\cos^2\psi + \sin^2\psi}$$

$$= 2 + \cos\psi$$

$$
\begin{aligned}
\text{Area(Torus)} &= \int_0^{2\pi}\int_0^{2\pi} \left| \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \psi} \right| \, d\theta \, d\psi \\
&= \int_0^{2\pi}\int_0^{2\pi} (2 + \cos\psi) \, d\theta \, d\psi \\
&= 2\pi \int_0^{2\pi} (2 + \cos\psi) \, d\psi \\
&= 8\pi^2
\end{aligned}
$$

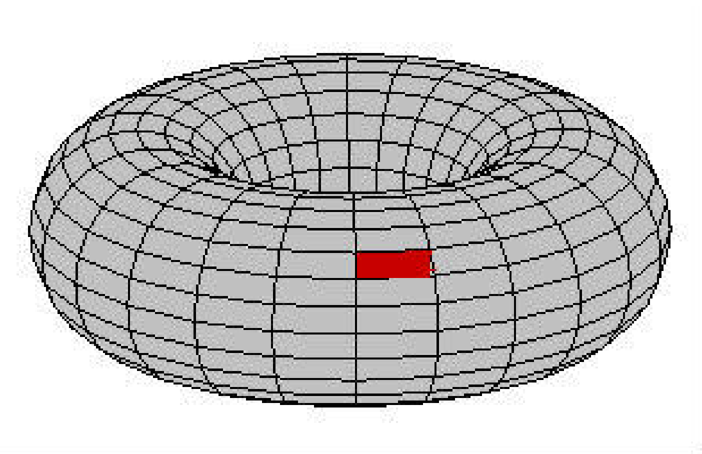
A donut is essentially a torus.

Suppose sugar is sprinkled over this donut and the sugar density (in mcg/cm^2) is:

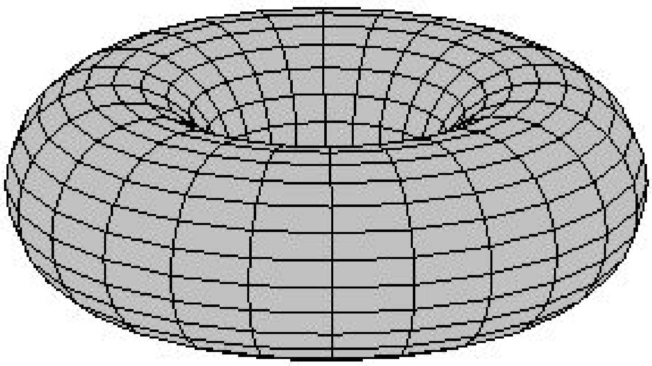
$$\delta = z \quad \text{for } z \geq 0$$

The density is 0 for $z < 0$

The mass of sugar on one section is $\delta \, dS$



The total mass on the donut is $\iint_S \delta \, dS$



$$
\begin{aligned}
\vec{\mathbf{r}} &= \langle x,\ y,\ z\rangle \\
&= \langle (2+\cos\psi)\cos\theta,\ (2+\cos\psi)\sin\theta,\ \sin\psi\rangle
\end{aligned}
$$

So, $\delta = z = \sin\psi$ for $0 \le \psi \le \pi$
We already know that $dS = (2+\cos\psi)\,d\theta\,d\psi$

$$
\begin{aligned}
\text{Mass} = \iint_S \delta\,dS &= \int_0^{\pi}\int_0^{2\pi} \sin\psi(2+\cos\psi)\,d\theta\,d\psi \\
&= 2\pi\int_0^{\pi} (2\sin\psi + \sin\psi\cos\psi)\,d\psi \\
&= 8\pi \text{ mcg of sugar}
\end{aligned}
$$