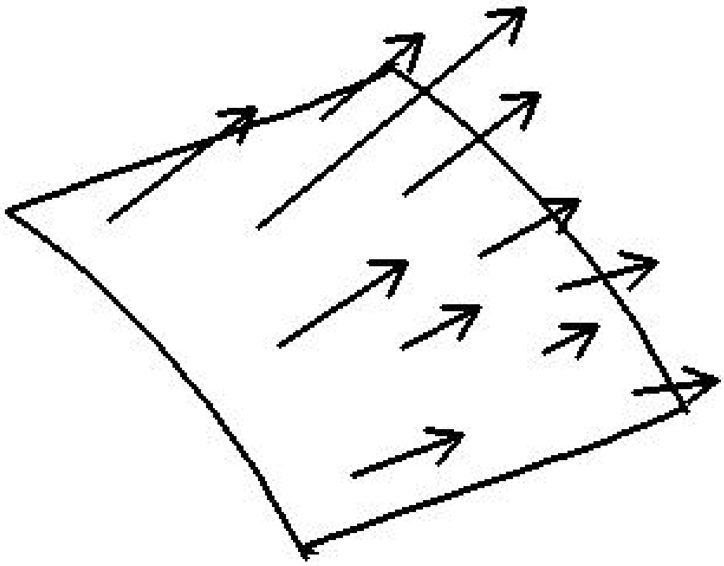
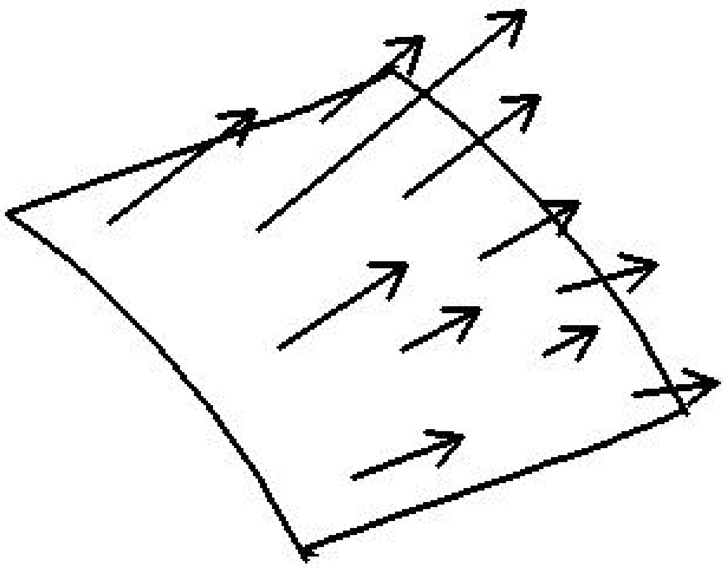
Surface Integrals

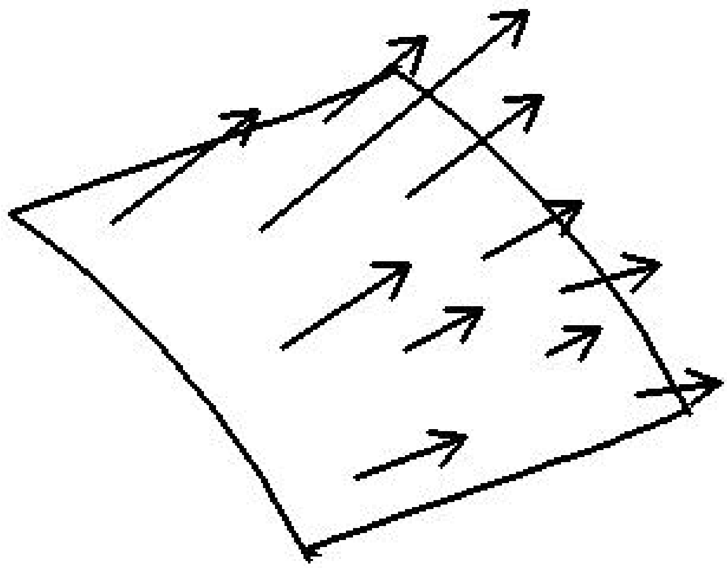
$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

Flow through a surface

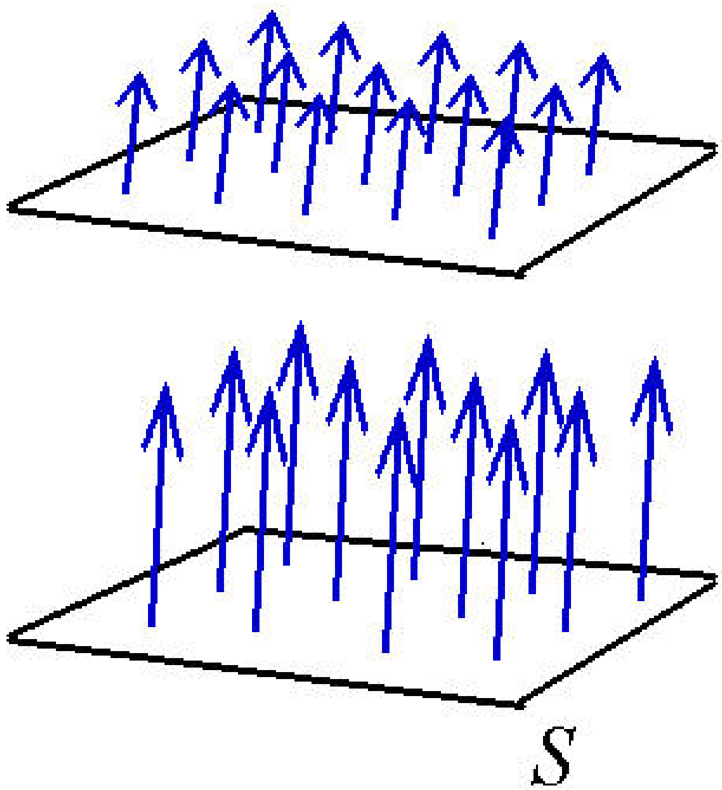


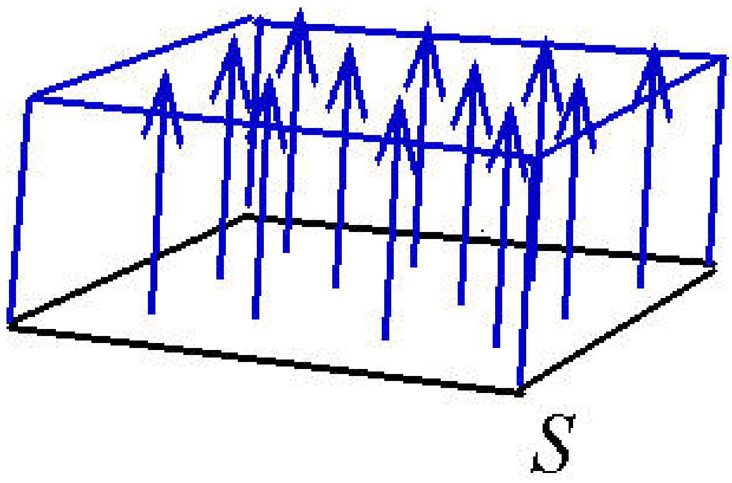


Flow of heat or energy (joules per sec per meter^2)



Flow of heat or energy (joules per sec per meter^2)
Flow of electric charge (amperes per meter^2)
Flow of a fluid (kg per sec per meter^2)


S



$$|\vec{\mathbf{v}}| \cdot \text{Area}(S) = \frac{\text{meters}}{\text{sec}} \cdot (\text{sq meters}) = \frac{\text{cubic meters}}{\text{sec}}$$

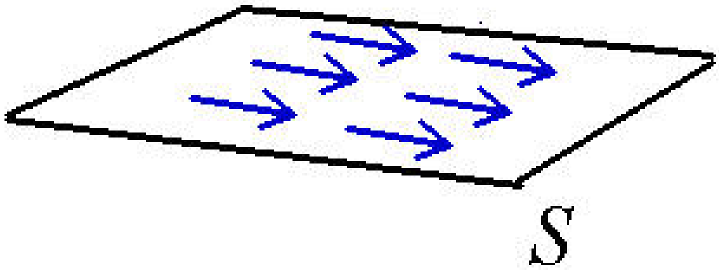
Let ρ is the density of this fluid (in kilograms per cubic meter)

$$\rho|\vec{\mathbf{v}}| \cdot \text{Area}(S)$$

$$\frac{\text{kilograms}}{\text{cubic meters}} \cdot \frac{\text{cubic meters}}{\text{second}} = \frac{\text{kilograms}}{\text{second}}$$

Let's define $\vec{\mathbf{F}} = \rho\vec{\mathbf{v}}$ so that $|\vec{\mathbf{F}}| \cdot \text{Area}(S)$ is the rate at which mass is flowing.

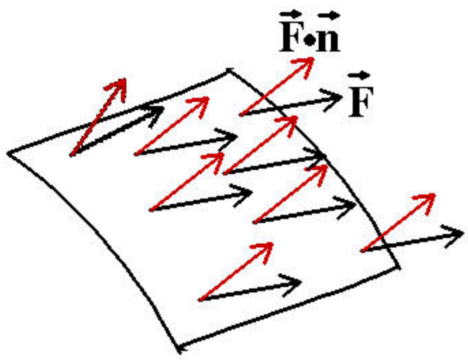
$\vec{\mathbf{F}}$ is not necessarily perpendicular to the surface.



If $\vec{\mathbf{n}}$ represents the *unit normal vector* to the surface S then $\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}$ is the normal component. and

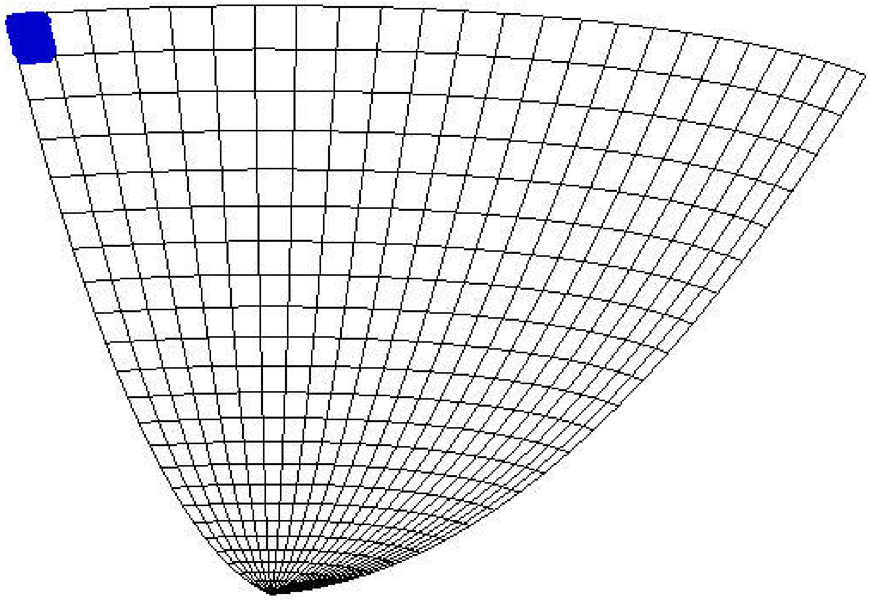
$$(\vec{\mathbf{F}} \bullet \vec{\mathbf{n}})(\text{Area}(S))$$

is the rate at which mass is flowing through the surface.



Divide S into many small sections where each section has area ΔS. The flux through each section is approximated by the expression:

$$\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, \Delta S$$



The flux through the entire surface S is the limit of the sum of terms of the form $\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, \Delta S$

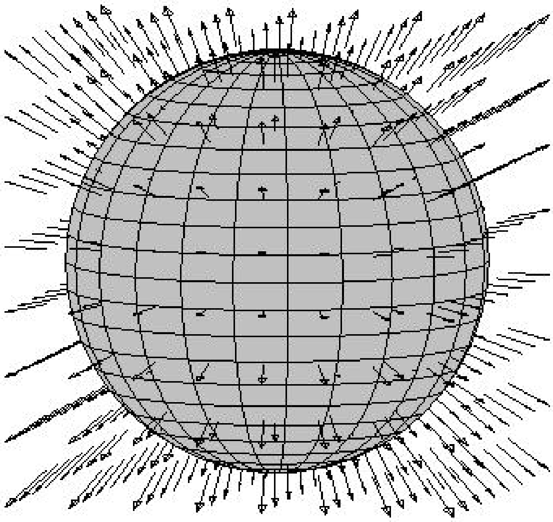
$$\Phi_S = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

Other notation: Let $\vec{\mathbf{dS}} = \vec{\mathbf{n}}\, dS$

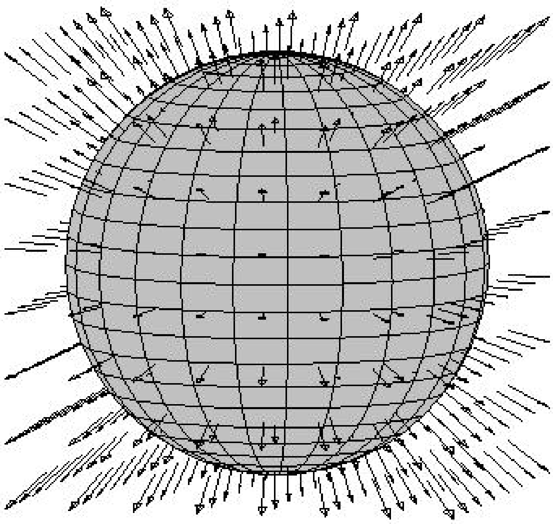
$$\Phi_S = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{dS}}$$

Closed Surfaces and Enclosed Volumes

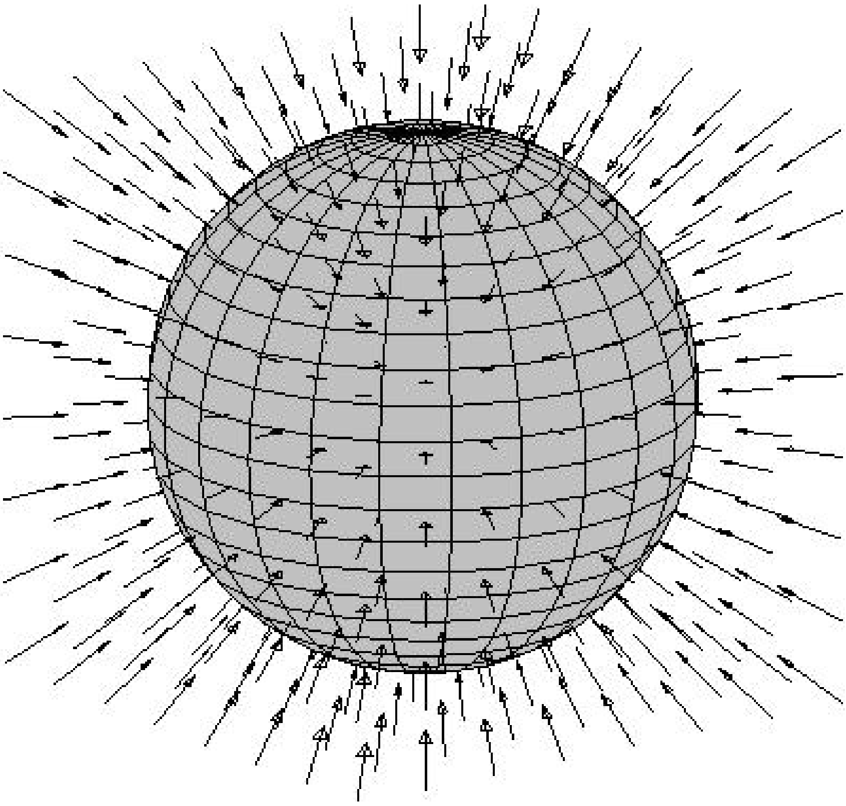
There are some surfaces, such as spheres, ellipsoids and cubes that have a well defined *inside* and *outside*. In these cases, we will take the direction of $\vec{\mathbf{n}}$ to be towards the outside.



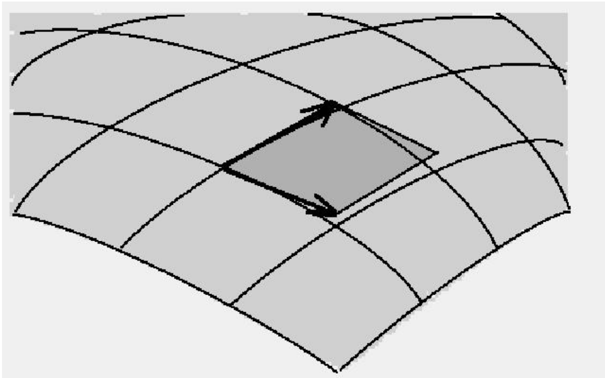
If $\vec{\mathbf{F}}$ points towards the outside, $\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}$ is positive



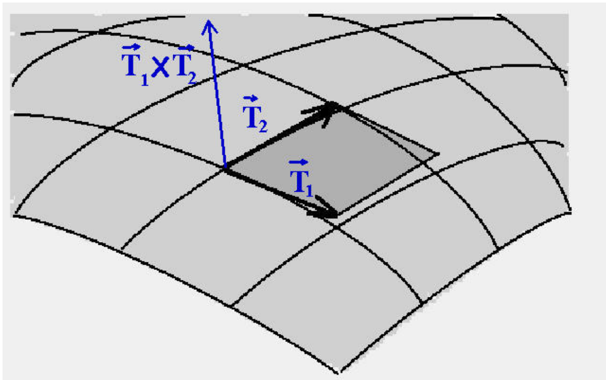
In the next example, $\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}$ is negative



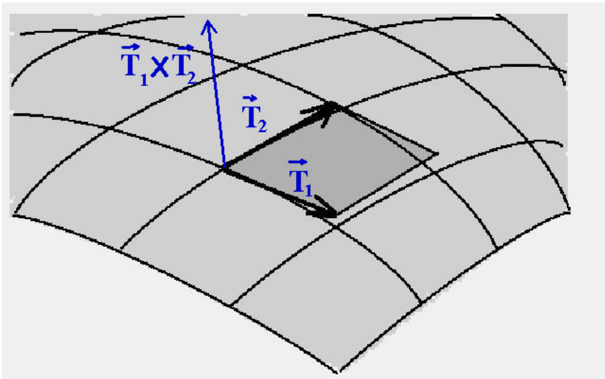
There is a convenient formula for $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$



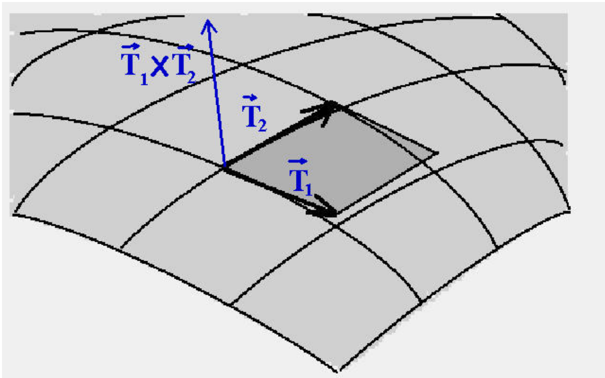
$$\vec{\mathbf{T}}_1 = \frac{\partial \vec{\mathbf{r}}}{\partial u} du \qquad \vec{\mathbf{T}}_2 = \frac{\partial \vec{\mathbf{r}}}{\partial v} dv$$



$$dS = |\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2|$$



$$\vec{\mathbf{n}} = \frac{\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2}{|\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2|}$$

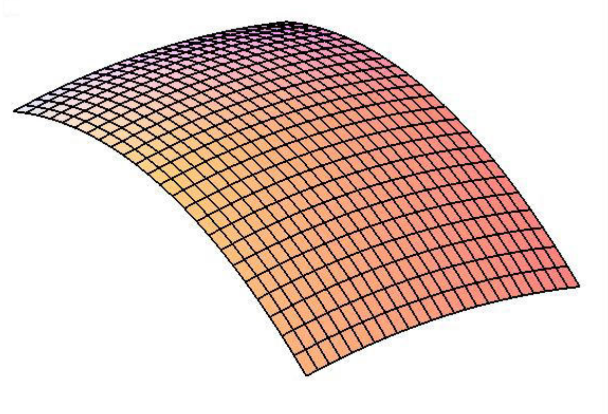


$$\vec{\mathbf{n}}\,dS = \frac{\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2}{|\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2|} \cdot |\vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2| = \vec{\mathbf{T}}_1 \times \vec{\mathbf{T}}_2$$

$$\text{Substitute } \vec{\mathbf{T}}_1 = \frac{\partial \vec{\mathbf{r}}}{\partial u} du \qquad \vec{\mathbf{T}}_2 = \frac{\partial \vec{\mathbf{r}}}{\partial v} dv$$

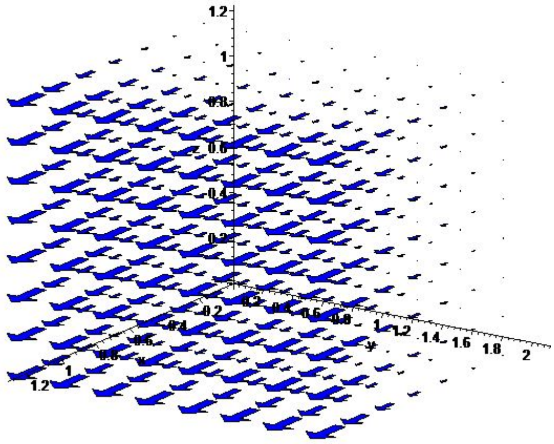
$$\vec{\mathbf{n}}\,dS = \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\,du\,dv$$

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} \vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right)\, du\, dv$$



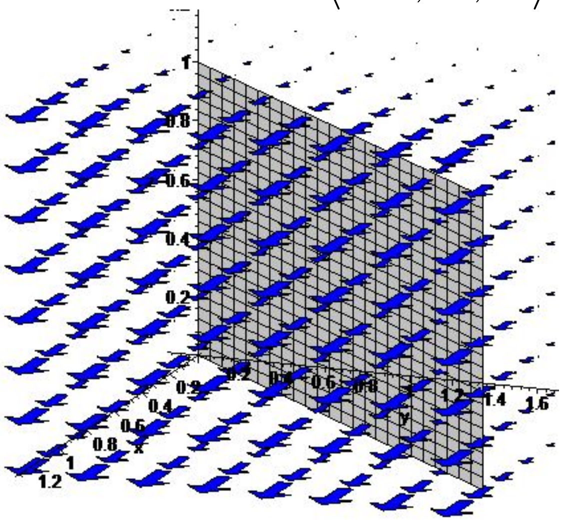
Example:

$$\vec{\mathbf{F}} = 3x^2\vec{\mathbf{i}} = \left\langle 3x^2,\ 0,\ 0 \right\rangle$$

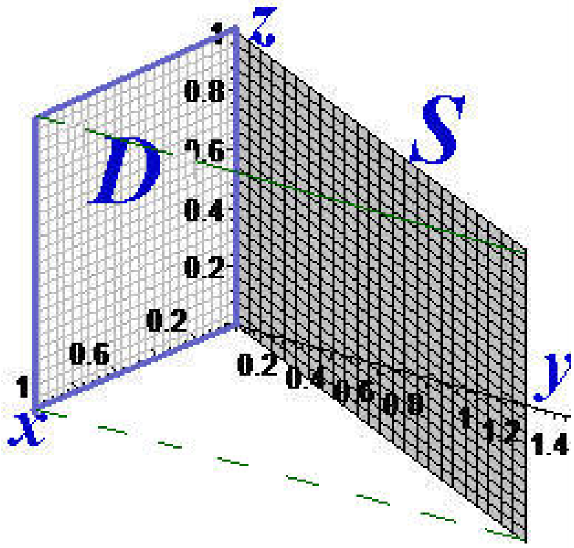


Example:

$$\vec{\mathbf{F}} = 3x^2\vec{\mathbf{i}} = \left\langle 3x^2,\ 0,\ 0 \right\rangle$$



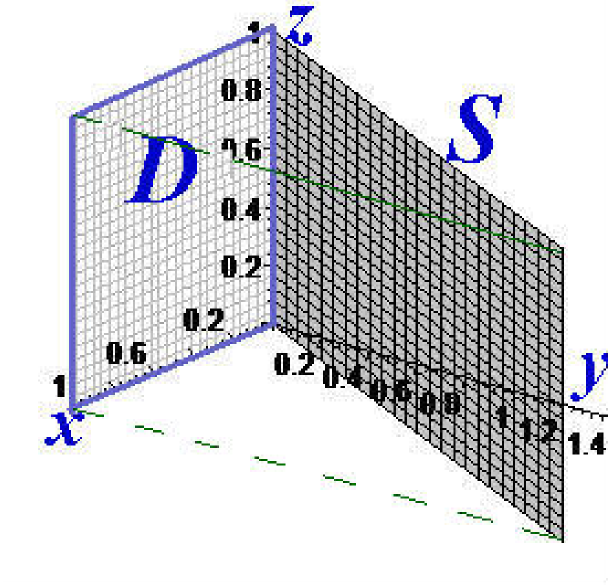
$$y = 2x$$



z
1
0.8
0.6
0.4
0.2
D
S
0.2
0.6
1
x
0.2
0.4
0.6
0.8
1
1.2
1.4
y

If every point on the surface satisfies the equation $y = 2x$ then:

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle x,\ 2x,\ z\rangle$$



In general,

$$\vec{\mathbf{n}}\,dS = \frac{\partial\vec{\mathbf{r}}}{\partial u} \times \frac{\partial\vec{\mathbf{r}}}{\partial v}\,du\,dv$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle x,\ 2x,\ z\rangle$$

$$\vec{\mathbf{n}}\,dS = \frac{\partial\vec{\mathbf{r}}}{\partial x} \times \frac{\partial\vec{\mathbf{r}}}{\partial z}\,dx\,dz$$

In general,

$$\vec{\mathbf{n}}\,dS = \frac{\partial\vec{\mathbf{r}}}{\partial u} \times \frac{\partial\vec{\mathbf{r}}}{\partial v}\,du\,dv$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle x,\ 2x,\ z\rangle$$

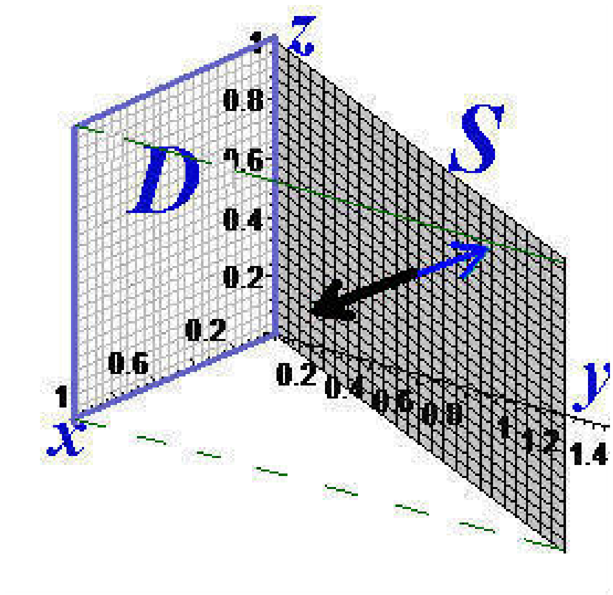
$$\vec{\mathbf{n}}\,dS = \frac{\partial\vec{\mathbf{r}}}{\partial x} \times \frac{\partial\vec{\mathbf{r}}}{\partial z}\,dx\,dz$$

or is it

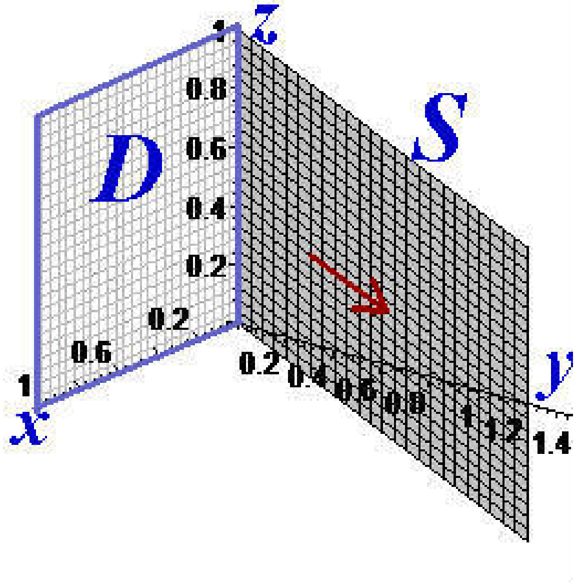
$$\vec{\mathbf{n}}\,dS = \frac{\partial\vec{\mathbf{r}}}{\partial z} \times \frac{\partial\vec{\mathbf{r}}}{\partial x}\,dx\,dz$$

?????

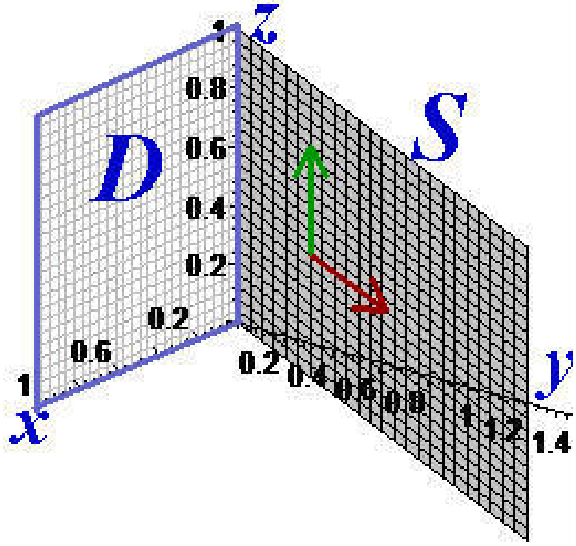
Which direction should we take for $\vec{\mathbf{n}}$?



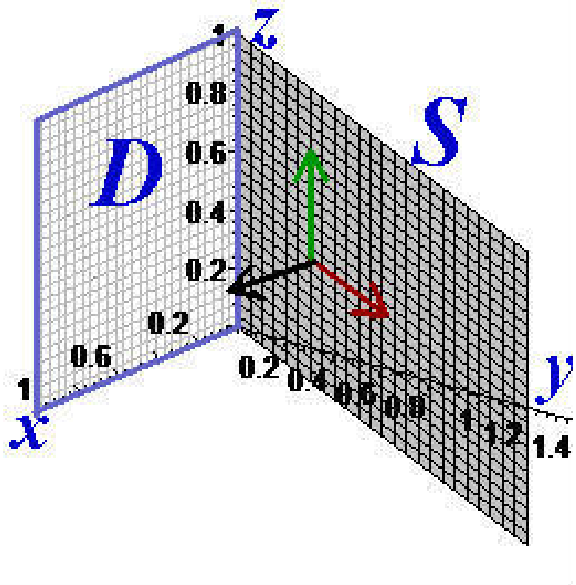
The red arrow is $\frac{\partial \vec{\mathbf{r}}}{\partial x}$



The red arrow is $\frac{\partial \vec{\mathbf{r}}}{\partial x}$. The green arrow is $\frac{\partial \vec{\mathbf{r}}}{\partial z}$



By the *right-hand rule*, the vector we want is $\frac{\partial\vec{\mathbf{r}}}{\partial x} \times \frac{\partial\vec{\mathbf{r}}}{\partial z}$



$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle x,\ 2x,\ z \rangle$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\vec{\mathbf{i}} - \vec{\mathbf{j}} = \langle 2,\ -1,\ 0 \rangle$$

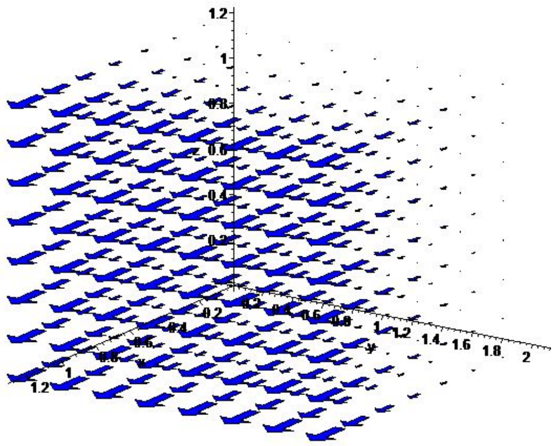
$$\vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} \right) = \langle 3x^2,\ 0,\ 0 \rangle \bullet \langle 2,\ -1,\ 0 \rangle = 6x^2$$

$$\iint_{\mathcal{D}} \vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} \right) \, dz \, dx = \int_0^1 \int_0^1 6x^2 \, dz \, dx$$

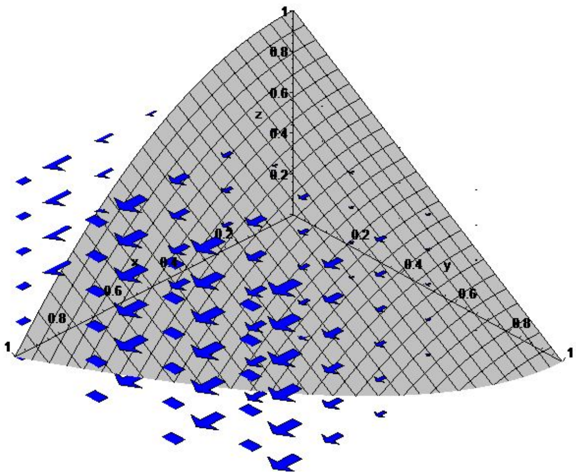
$$= 2$$

Example: Take the same vector field as before.

$$\vec{\mathbf{F}} = 3x^2\vec{\mathbf{i}}$$



Let Ω be the portion of $z = 1 - x^2 - y$ in the first octant.



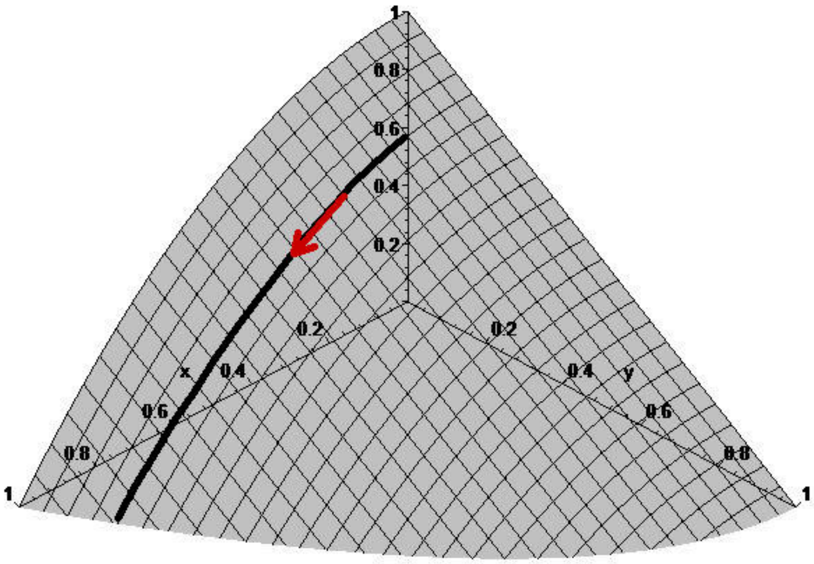
If $z = 1 - x^2 - y$ then $\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle x,\ y,\ 1 - x^2 - y\rangle$

$$\frac{\partial \vec{\mathbf{r}}}{\partial x} = \langle 1,\ 0,\ -2x\rangle$$

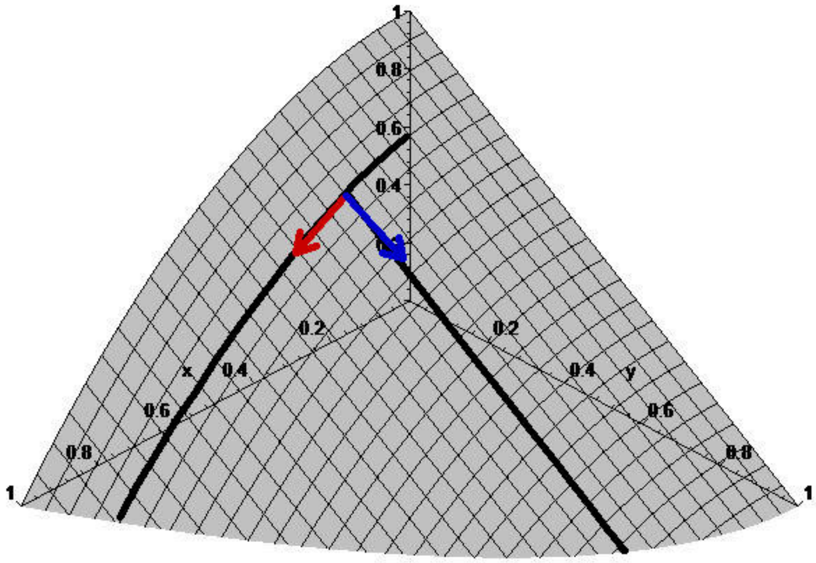
$$\frac{\partial \vec{\mathbf{r}}}{\partial y} = \langle 0,\ 1,\ -1\rangle$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & -2x \\ 0 & 1 & -1 \end{vmatrix} = \langle 2x,\ 1,\ 1\rangle$$

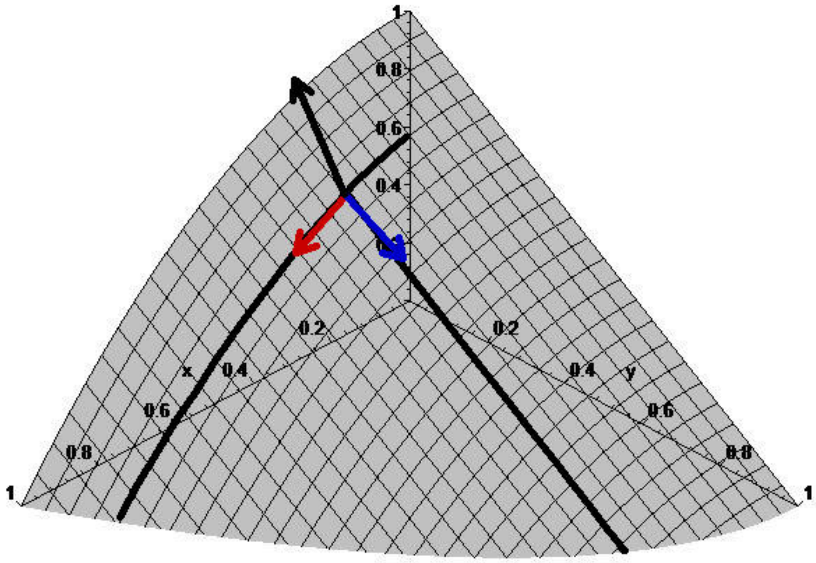
The red arrow is $\frac{\partial \vec{\mathbf{r}}}{\partial x}$.



The red arrow is $\frac{\partial \vec{\mathbf{r}}}{\partial x}$. The blue arrow is $\frac{\partial \vec{\mathbf{r}}}{\partial y}$

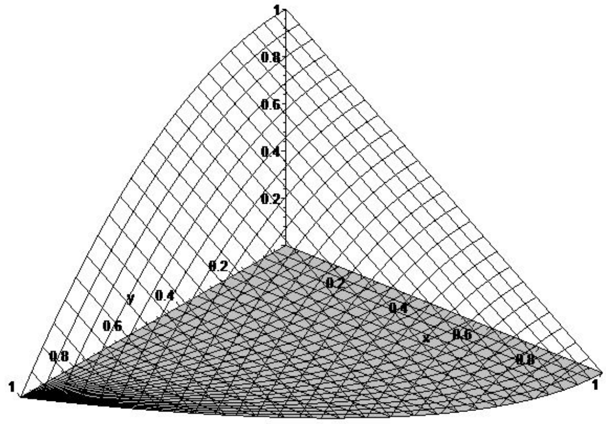


$\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y}$ will point in the direction we want for $\vec{\mathbf{n}}$



$$\vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y} \right) = \langle 3x^2, \ 0, \ 0 \rangle \bullet \langle 2x, \ 1, \ 1 \rangle = 6x^3$$

$$\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}} \, dS = \iint_{\mathcal{D}} 6x^3 \, dy \, dx$$



$$\vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y} \right) = \langle 3x^2,\ 0,\ 0 \rangle \bullet \langle 2x,\ 1,\ 1 \rangle = 6x^3$$

$$\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} 6x^3\, dy\, dx = \int_0^1 \int_0^{1-x^2} 6x^3\, dy\, dx = \frac{1}{2}$$