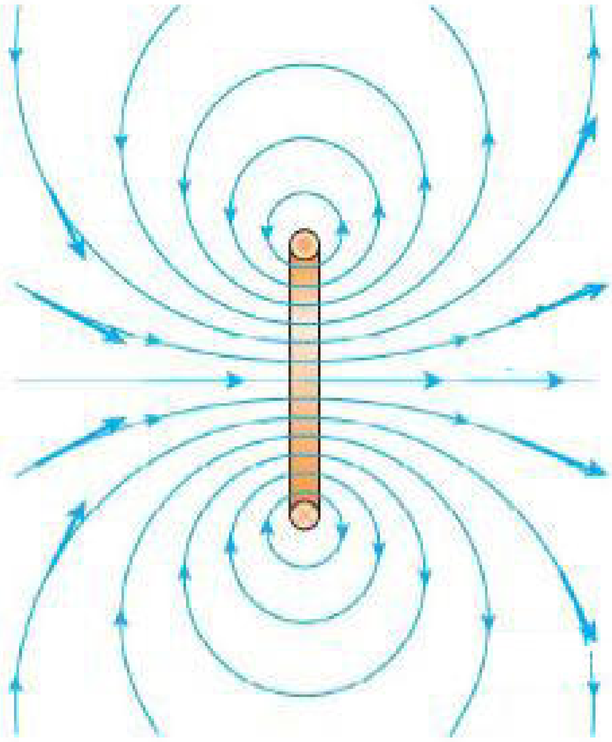
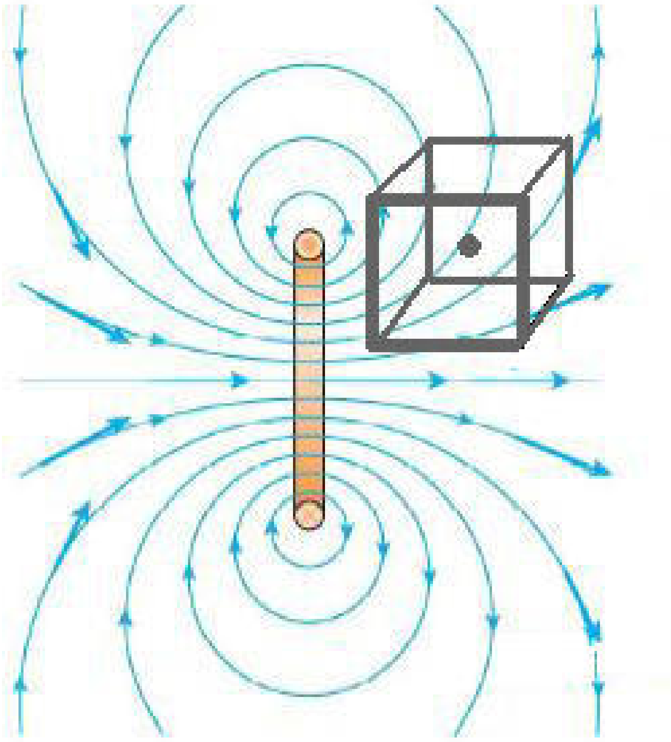
Surface Integrals Over Closed Surfaces
Dr. Elliott Jacobs

$$\oiint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

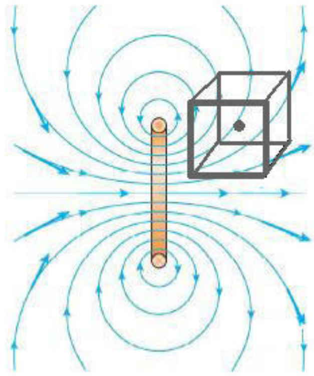
Magnetic Field



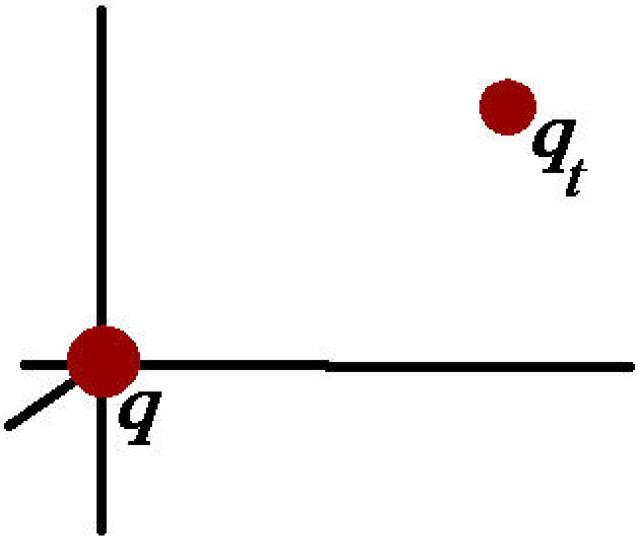
Magnetic Field



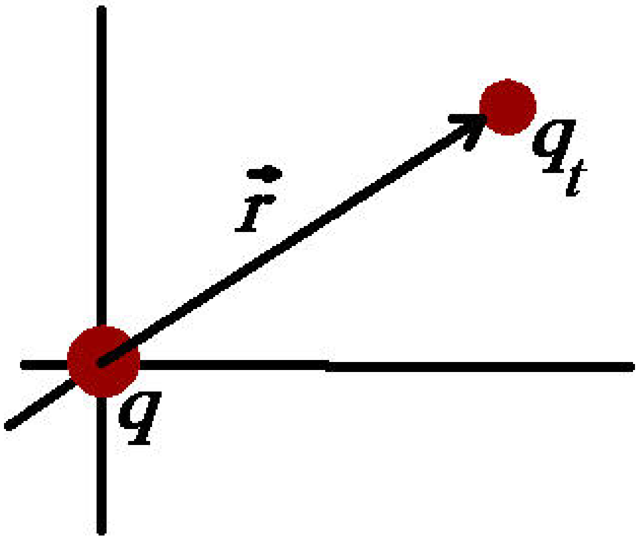
$$\iint_S \vec{\mathbf{B}} \bullet \vec{\mathbf{n}} \, dS = 0$$



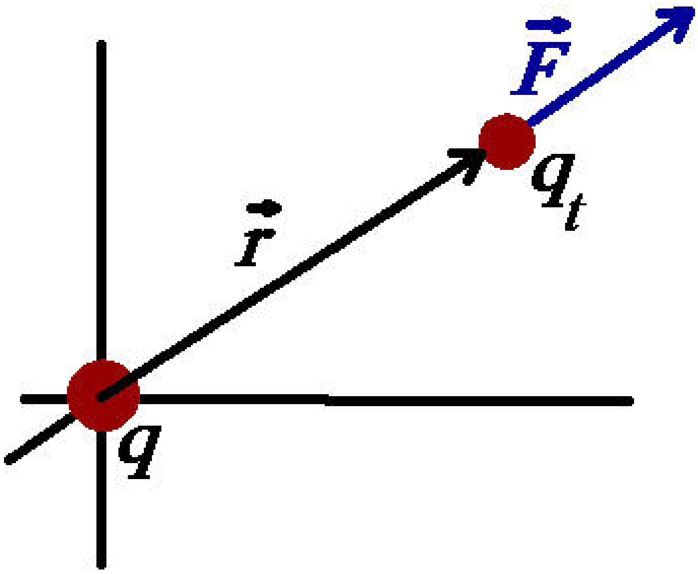
Two Positive Electric Charges



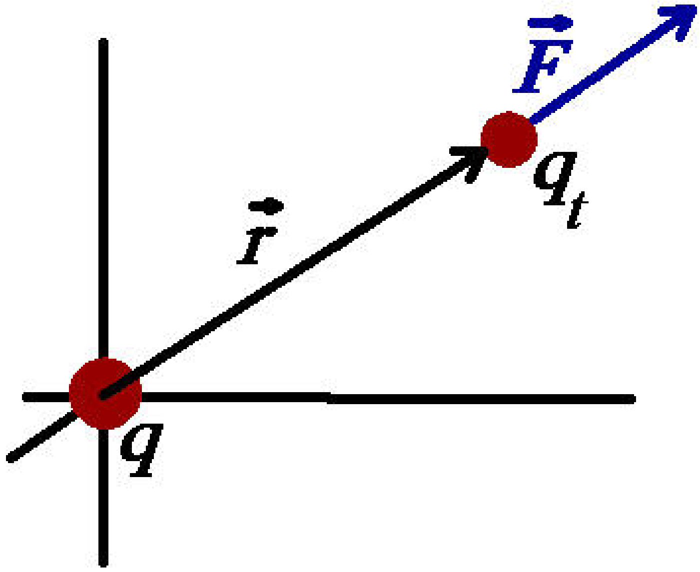
Two Positive Electric Charges



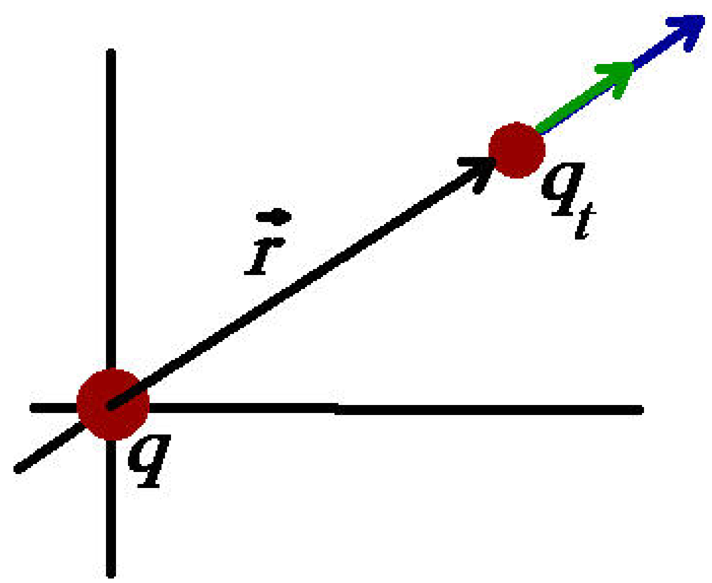
The electric force is in the same direction as $\vec{\mathbf{r}}$



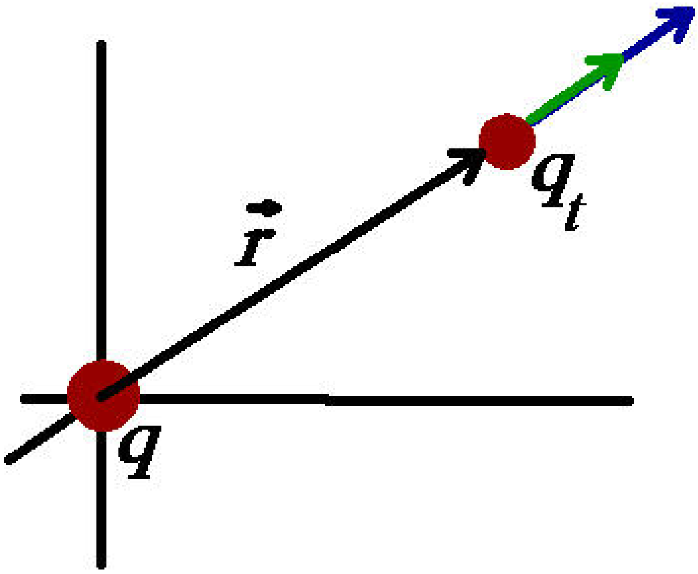
Coulomb's Law: $|\vec{\mathbf{F}}| = \frac{kqq_t}{|\vec{\mathbf{r}}|^2}$



The unit vector in the direction of $\vec{\mathbf{F}}$ is $\frac{\vec{\mathbf{F}}}{|\vec{\mathbf{F}}|}$



The unit vector in the direction of $\vec{\mathbf{r}}$ is $\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$



$$\frac{\vec{\mathbf{F}}}{|\vec{\mathbf{F}}|} = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$$

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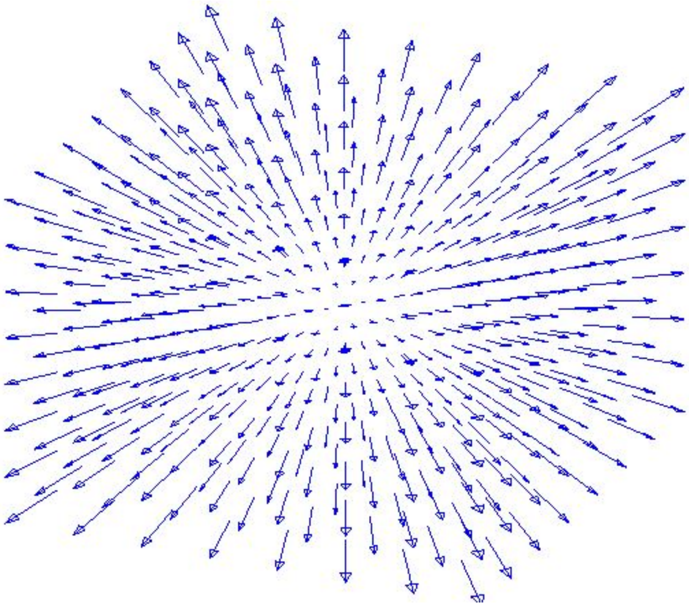
$$\vec{\mathbf{F}} = |\vec{\mathbf{F}}| \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = \frac{kqq_t}{|\vec{\mathbf{r}}|^2} \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = \frac{kqq_t}{|\vec{\mathbf{r}}|^3} \vec{\mathbf{r}}$$

$$\vec{\mathbf{F}} = \frac{kqq_t}{|\vec{\mathbf{r}}|^3}\vec{\mathbf{r}}$$

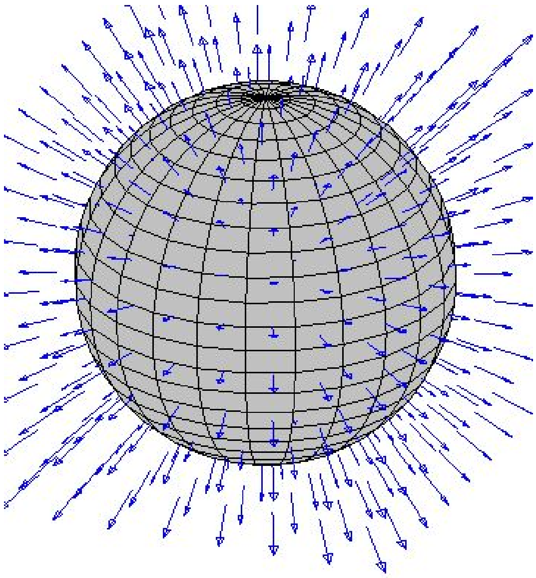
The electric field $\vec{\mathbf{E}}$ is the force per test charge

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_t} = \frac{kq}{|\vec{\mathbf{r}}|^3}\vec{\mathbf{r}}$$

$$\vec{\mathbf{E}} = \frac{kq}{|\vec{\mathbf{r}}|^3}\vec{\mathbf{r}}$$

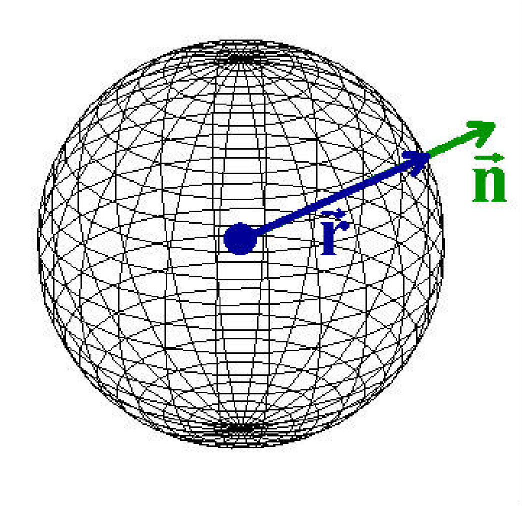


$$\iint_S \vec{\mathbf{E}} \bullet \vec{\mathbf{n}}\, dS$$



The unit normal vector $\vec{\mathbf{n}}$ points in the same direction as the position vector $\vec{\mathbf{r}}$

$$\vec{\mathbf{n}} = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$$



$$\vec{\mathbf{E}} \bullet \vec{\mathbf{n}} = \frac{kq\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3} \bullet \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = \frac{kq}{|\vec{\mathbf{r}}|^3} \frac{|\vec{\mathbf{r}}|^2}{|\vec{\mathbf{r}}|}$$

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$$= \frac{kq}{|\vec{\mathbf{r}}|^2}$$

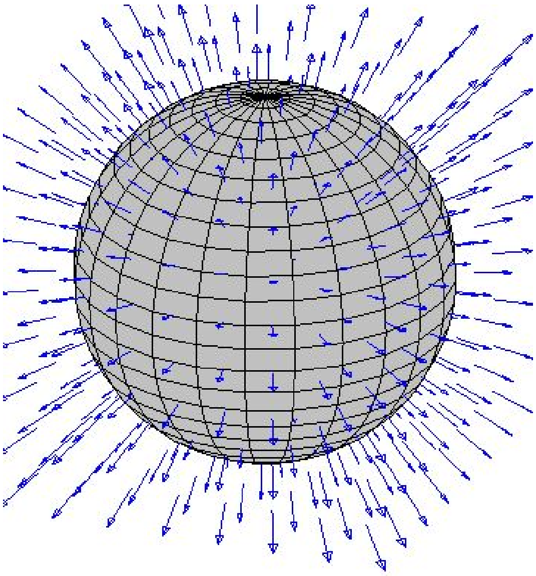
$$
\begin{aligned}
\vec{\mathbf{E}} \bullet \vec{\mathbf{n}} &= \frac{kq\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3} \bullet \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = \frac{kq}{|\vec{\mathbf{r}}|^3} \frac{|\vec{\mathbf{r}}|^2}{|\vec{\mathbf{r}}|} \\
&= \frac{kq}{|\vec{\mathbf{r}}|^2} \\
&= \frac{kq}{a^2}
\end{aligned}
$$

$$\iint_S \vec{\mathbf{E}} \bullet \vec{\mathbf{n}}\, dS = \iint_S \frac{kq}{a^2}\, dS = \frac{kq}{a^2} \iint_S dS$$

$$
\begin{aligned}
\iint_S \vec{\mathbf{E}} \bullet \vec{\mathbf{n}} \, dS &= \iint_S \frac{kq}{a^2} \, dS = \frac{kq}{a^2} \iint_S dS \\
&= \frac{kq}{a^2} \cdot \text{Area}(S) \\
&= \frac{kq}{a^2} \cdot 4\pi a^2 \\
&= 4\pi kq
\end{aligned}
$$

Gauss's Law for Electric Fields:

$$\iint_S \vec{\mathbf{E}} \bullet \vec{\mathbf{n}}\, dS = 4\pi k q$$



Gauss's Law for Electric Fields:

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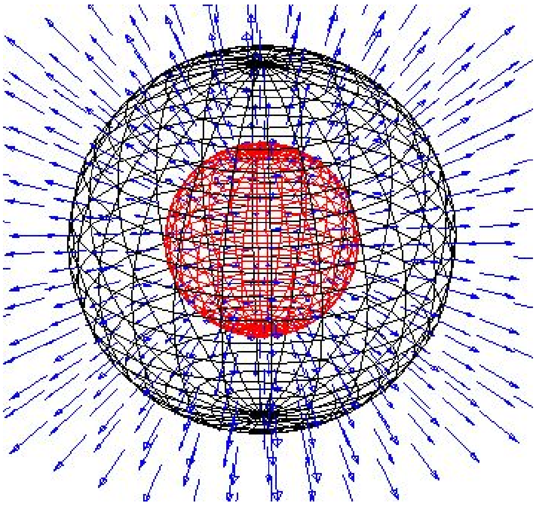
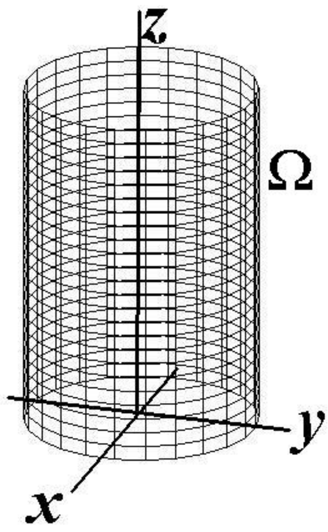
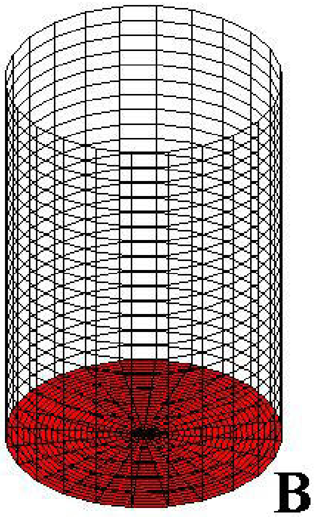


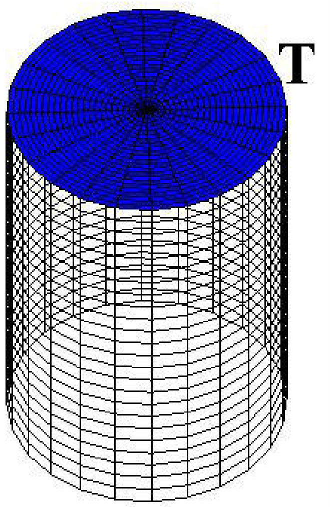
Let Ω be the surface forming the vertical side of a cylinder of radius 1 around the z axis, for $0 \leq z \leq 3$



Let B be the surface that forms the base of this cylinder. B is a disk of radius 1 in the xy plane.

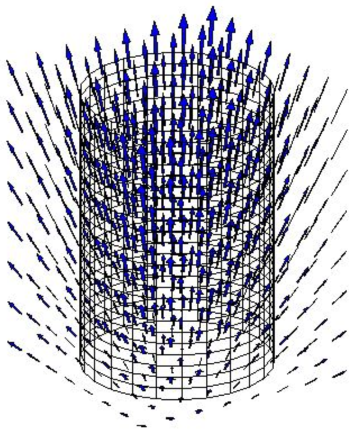


Let T be the surface that forms the top of this cylinder. T is a disk of radius 1 that is 3 units above the xy plane.



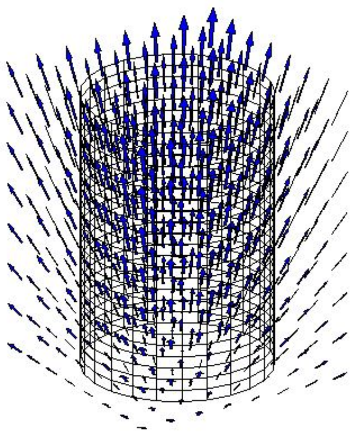
Let S be the closed surface around the cylinder
Let $\vec{\mathbf{F}} = \langle x,\ y,\ z+1 \rangle$.

Calculate the surface integral: $$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$



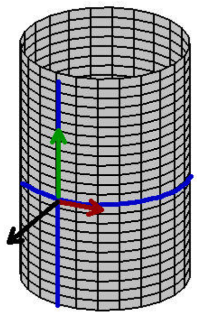
Calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ by adding up the flux through all three surfaces that form S

$$\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$



Start with Ω. $\quad \vec{\mathbf{r}} = \langle \cos\theta,\ \sin\theta,\ z\rangle$

$$\frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta,\ \sin\theta,\ 0\rangle$$



$$\frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta,\ \sin\theta,\ 0 \rangle$$

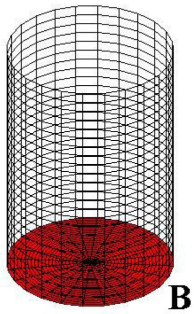
$$\vec{\mathbf{F}} = \langle x,\ y,\ z+1 \rangle = \langle \cos\theta,\ \sin\theta,\ z+1 \rangle$$

$$\vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} \right) = \cos^2\theta + \sin^2\theta = 1$$

$$\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \int_0^{2\pi} \int_0^3 1\, dz\, d\theta = 6\pi$$

Next, integrate over the bottom B

$$\vec{\mathbf{r}} = \langle x,\ y,\ 0\rangle \quad \text{so} \quad \frac{\partial \vec{\mathbf{r}}}{\partial y} \times \frac{\partial \vec{\mathbf{r}}}{\partial x} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\vec{\mathbf{k}}$$



$$\vec{\mathbf{r}} = \langle x,\ y,\ 0\rangle \quad \text{so} \quad \frac{\partial \vec{\mathbf{r}}}{\partial y} \times \frac{\partial \vec{\mathbf{r}}}{\partial x} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\vec{\mathbf{k}}$$

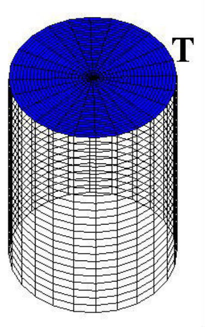
$$\vec{\mathbf{F}} = \langle x,\ y,\ z+1\rangle = \langle x,\ y,\ 0+1\rangle$$

$$\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \langle x,\ y,\ 1\rangle \bullet \langle 0,\ 0,\ -1\rangle\, dS = -1\, dS$$

$$\iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = (-1) \iint_B dS = -\text{Area}(B) = -\pi$$

Next, integrate over T

$$\vec{\mathbf{r}} = \langle x,\ y,\ 3\rangle \quad \text{so} \quad \frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{\mathbf{k}}$$



$$\vec{\mathbf{r}} = \langle x,\ y,\ 3\rangle \quad \text{so} \quad \frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{\mathbf{k}}$$

$$\vec{\mathbf{F}} = \langle x,\ y,\ z+1\rangle = \langle x,\ y,\ 3+1\rangle$$

$$\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \langle x,\ y,\ 4\rangle \bullet \langle 0,\ 0,\ 1\rangle\, dS = 4\, dS$$

$$\iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = 4 \iint_T dS = 4\text{Area}(T) = 4\pi$$

Finally, add all three surface integrals together:

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = 6\pi - \pi + 4\pi = 9\pi$$