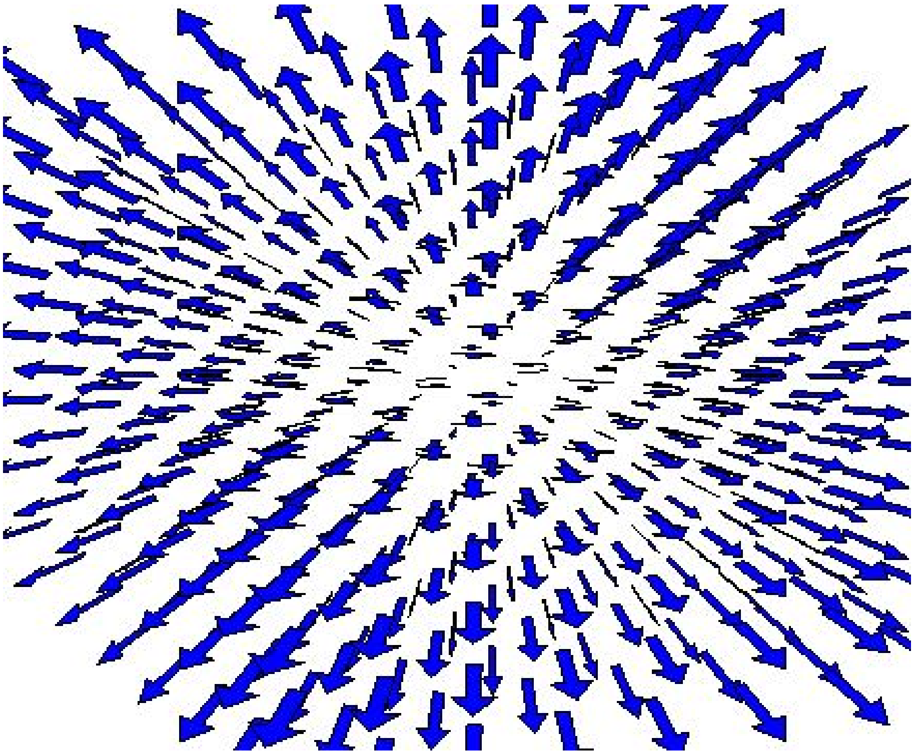
Divergence of a Vector Field

Dr. Elliott Jacobs



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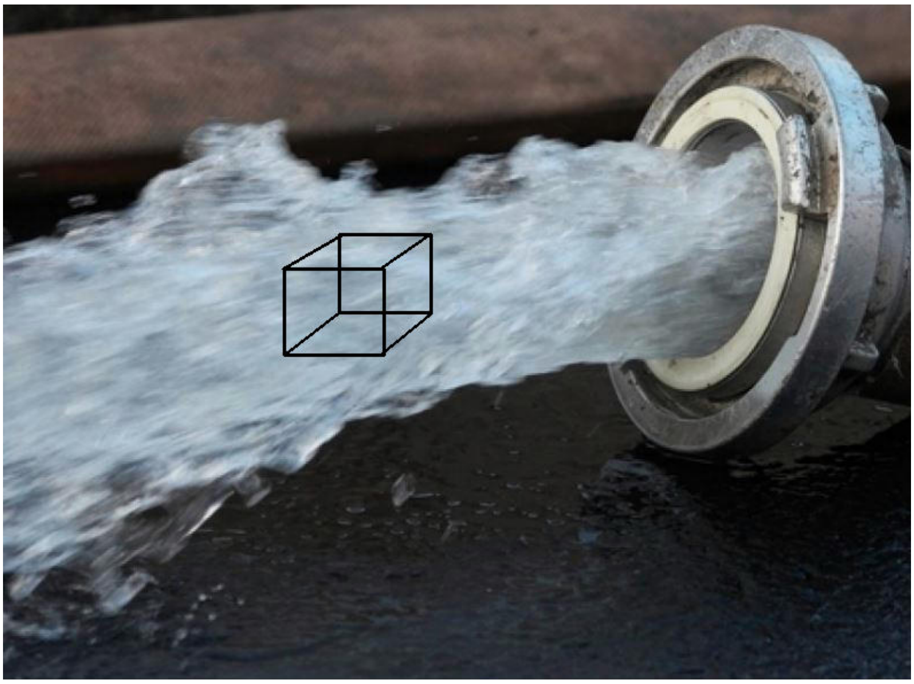
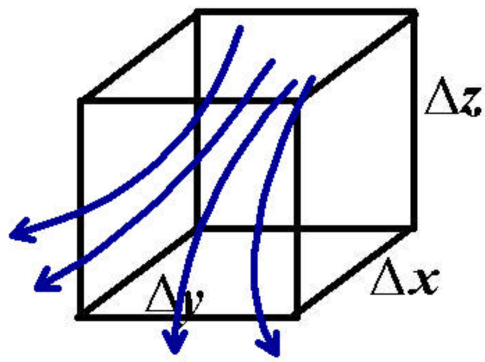
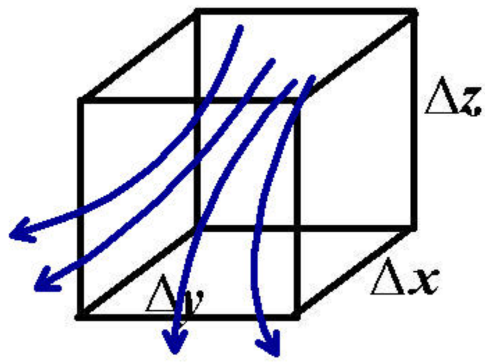


$$\Phi_S = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$



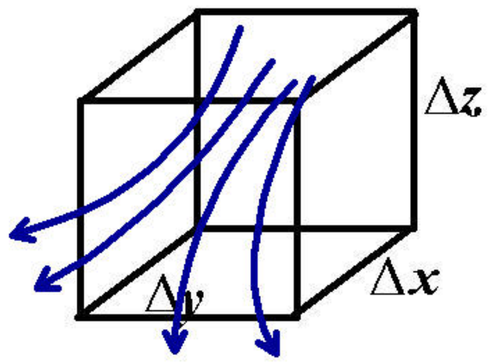
Let $M(t)$ denote mass inside the cube at time t.

$$|\Phi_S| = \left|\frac{dM}{dt}\right|$$



Let $M(t)$ denote mass inside the cube at time t.

$$\Phi_S = -\frac{dM}{dt}$$



Let ρ be the density (in kilograms per meter^3) at location (x, y, z) at time t seconds. If V denotes the interior of the solid then

$$M(t) = \iiint_V \rho \, dV$$

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If $\mathrm{vol}(S) \approx 0$ then the mass is approximated by:

$$M(t) = \iiint_V \rho \, dV \approx \rho \iiint_V \, dV = \rho \, \mathrm{vol}(V)$$

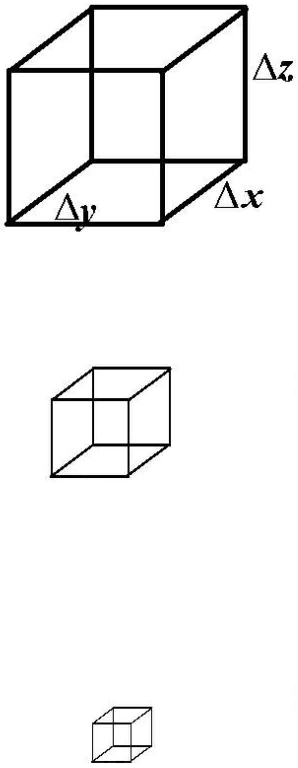
$$M(t) \approx \rho \, \mathrm{vol}(V)$$

Therefore, the flux is approximated by:

$$\Phi_S = -\frac{dM}{dt} \approx -\frac{\partial \rho}{\partial t} \, \mathrm{vol}(V)$$

$$\frac{\Phi_S}{\mathrm{vol}(V)} \approx -\frac{\partial \rho}{\partial t}$$

This approximation improves as $\mathrm{vol}(V) \longrightarrow 0$



In the limit,

$$\lim_{\mathrm{vol}(V)\to 0} \frac{\Phi_S}{\mathrm{vol}(V)} = -\frac{\partial \rho}{\partial t}$$

This is the **divergence** of the vector field $\vec{\mathbf{F}}$.

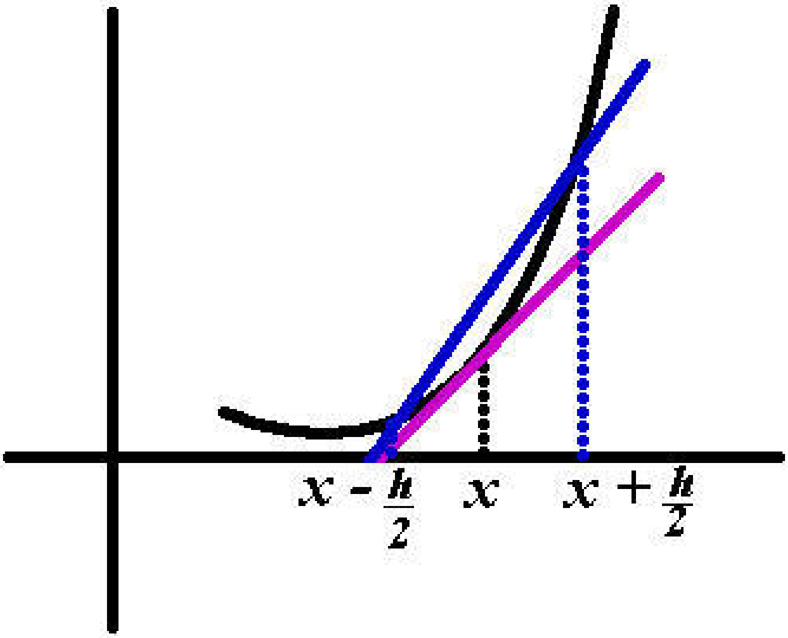
$$\operatorname{div} \vec{\mathbf{F}} = -\frac{\partial \rho}{\partial t}$$

Partial Derivative Formula for div $\vec{\mathbf{F}}$

Recall that the derivative of $f(x)$ is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}{h}$$



$$f'(x) = \lim_{h \to 0} \frac{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}{h}$$

$$\frac{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}{h} \approx f'(x)$$

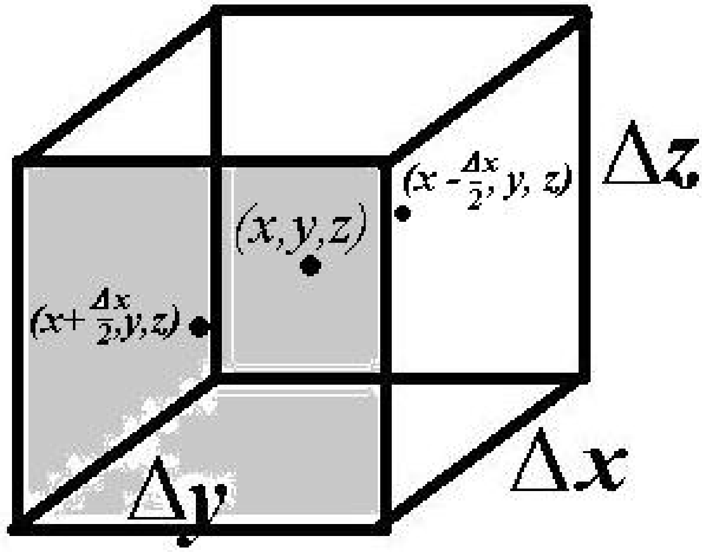
where the approximation improves as $h \to 0$

We can do exactly the same sort of thing for partial derivatives. If $f = f(x, y, z)$ then

$$\frac{f\left(x + \frac{\Delta x}{2},\ y,\ z\right) - f\left(x - \frac{\Delta x}{2},\ y,\ z\right)}{\Delta x} \approx \frac{\partial f}{\partial x}$$

where the error in approximation $\to 0$ as $\Delta x \to 0$

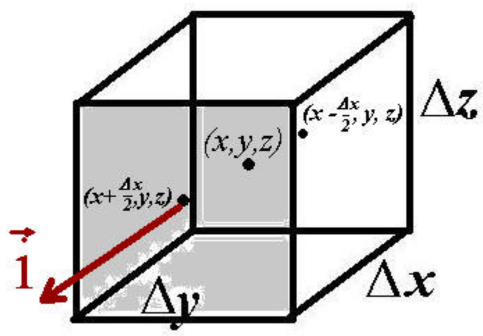
Consider the flux through the following cube:



On the front portion, $\vec{\mathbf{n}} = \vec{\mathbf{i}}$

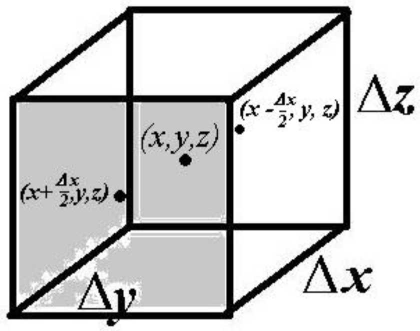
$\vec{\mathbf{F}} \bullet \vec{\mathbf{n}} = \langle F_1,\ F_2,\ F_3 \rangle \bullet \langle 1,\ 0,\ 0 \rangle = F_1$

$$\iint_{S_1} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{S_1} F_1\, dS$$



If this is a small cube, then the flux through the front may be approximated at $\left(x + \frac{\Delta x}{2},\ y,\ z\right)$

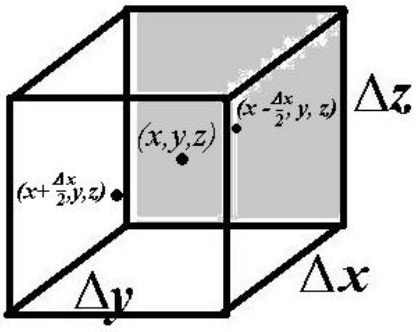
$$\iint_{S_1} F_1\, dy\, dz \ \approx F_1\left(x + \frac{\Delta x}{2},\ y,\ z\right) \Delta y\, \Delta z$$



Approximation of the flux through the front S_1.

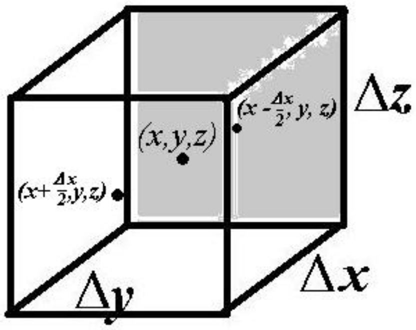
$$\iint_{S_1} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \approx F_1\left(x + \frac{\Delta x}{2},\ y,\ z\right) \Delta y\, \Delta z$$

Next, approximate the flux through the back S_2



Flux through the back:

$$\iint_{S_2} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \approx -F_1\left(x - \frac{\Delta x}{2},\ y,\ z\right) \Delta y\, \Delta z$$



Total approximate flux through the front and back:

$$\left(F_1\left(x+\frac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\frac{\Delta x}{2},\ y,\ z\right)\right)\Delta y\,\Delta z$$

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$$\left(F_1\left(x+\tfrac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\tfrac{\Delta x}{2},\ y,\ z\right)\right)\Delta y\,\Delta z$$

This is equal to:

$$\frac{F_1\left(x+\tfrac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\tfrac{\Delta x}{2},\ y,\ z\right)}{\Delta x}\cdot\Delta x\,\Delta y\,\Delta z$$

Total approximate flux through the front and back:

$\left(F_1\left(x+\frac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\frac{\Delta x}{2},\ y,\ z\right)\right)\Delta y\,\Delta z$

This is equal to:

$$\frac{F_1\left(x+\frac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\frac{\Delta x}{2},\ y,\ z\right)}{\Delta x}\cdot\Delta x\,\Delta y\,\Delta z$$

This is approximately equal to:

$$\frac{\partial F_1}{\partial x}(x,y,z)\Delta x\,\Delta y\,\Delta z$$

Flux approximation through the front and back:

$$\frac{\partial F_1}{\partial x}\mathrm{Vol}(V)$$

Flux approximation through the front and back:

$$\frac{\partial F_1}{\partial x}\text{Vol}(V)$$

Through left and right sides:

$$\frac{\partial F_2}{\partial y}\text{Vol}(V)$$

Through top and bottom:

$$\frac{\partial F_3}{\partial z}\text{Vol}(V)$$

If we combine these quantities, we get the approximation of the flux through the entire surface S surrounding the cube:

$$\Phi_S \approx \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \mathrm{Vol}(V)$$

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$$\Phi_S \approx \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \mathrm{Vol}(V)$$

The approximation becomes better and better as the volume shrinks to 0.

$$\operatorname{div} \vec{\mathbf{F}} = \lim_{vol(V) \to 0} \frac{\Phi_S}{\mathrm{Vol}(V)} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

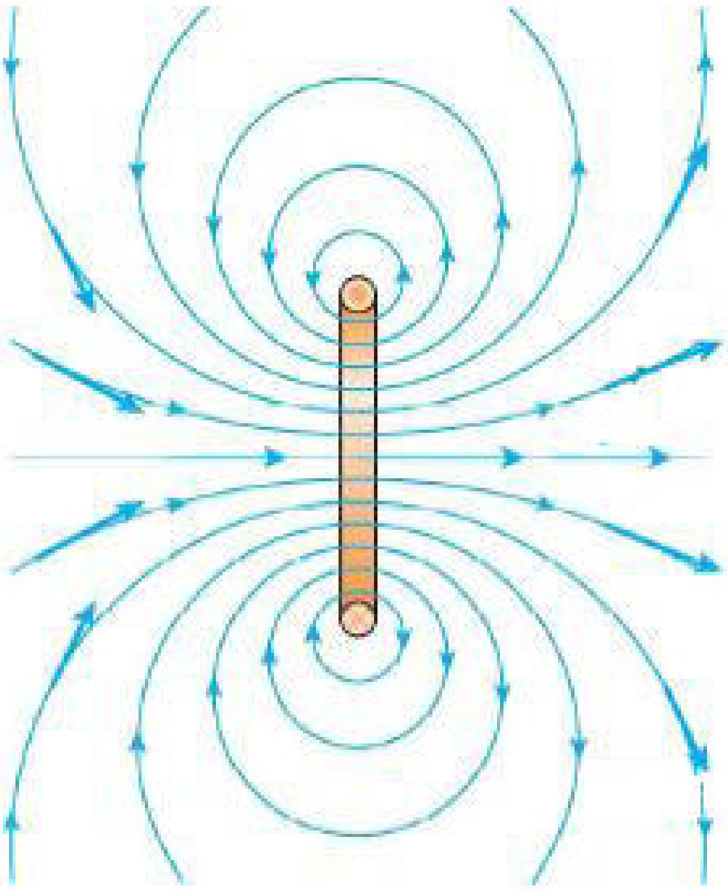
Example:

Let $\vec{\mathbf{F}} = \langle xy,\ z^2 \sin x,\ e^z x \rangle$.

$$F_1 = xy \qquad F_2 = z^2 \sin x \qquad F_3 = e^z x$$

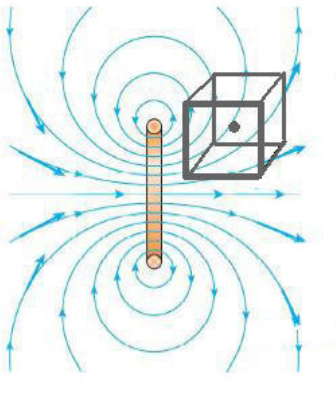
$$\text{div } \vec{\mathbf{F}} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(z^2 \sin x\right) + \frac{\partial}{\partial z}\left(e^z x\right) = y + e^z x$$

$$\begin{aligned}
\operatorname{div}\vec{\mathbf{F}} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\
&= \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \langle F_1,\ F_2,\ F_3 \rangle \\
&= \nabla \bullet \vec{\mathbf{F}}
\end{aligned}$$



Gauss's Law for Magnetism:

$$\Phi_S = \iint_S \vec{\mathbf{B}} \bullet \vec{\mathbf{n}}\, dS = 0$$



$$\Phi_S = \iint_S \vec{\mathbf{B}} \bullet \vec{\mathbf{n}}\, dS = 0$$

$$\frac{\Phi_S}{\mathrm{vol}(V)} = 0$$

Let $\mathrm{vol}(V) \to 0$

$$\nabla \bullet \vec{\mathbf{B}} = 0$$

$$\iint_S \vec{\mathbf{B}} \bullet \vec{\mathbf{n}}\, dS = 0$$

$$\nabla \bullet \vec{\mathbf{B}} = 0$$