The Divergence Theorem

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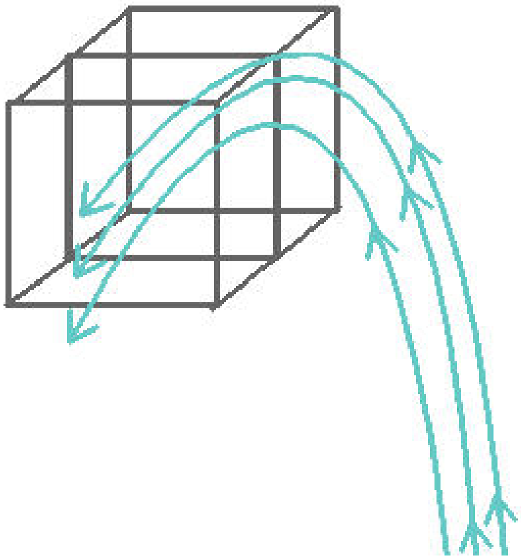
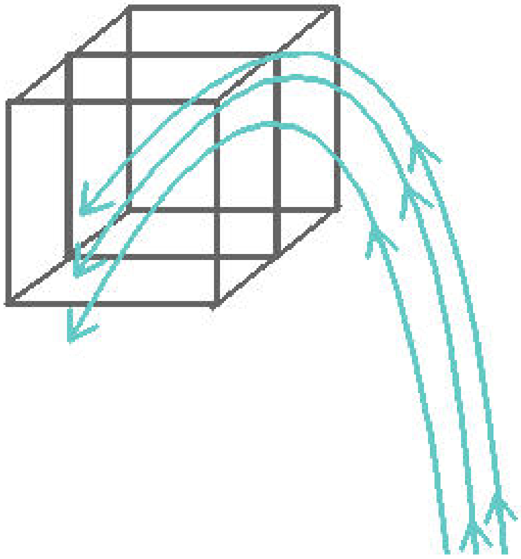
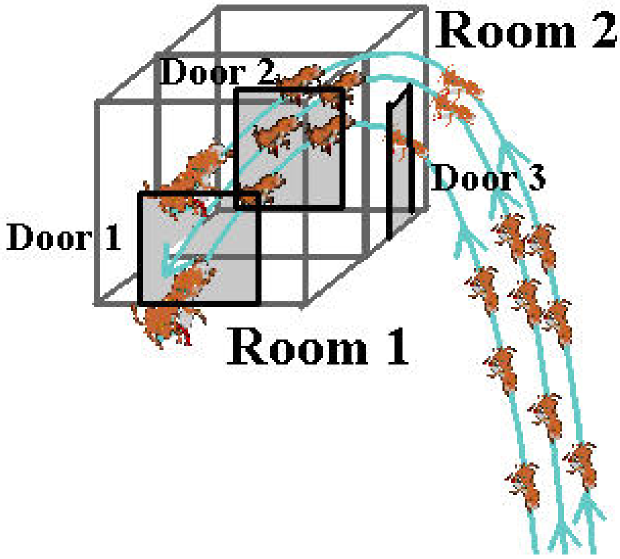


$\vec{\mathbf{F}}$ represents flow through a solid. Divide the solid into two compartments.

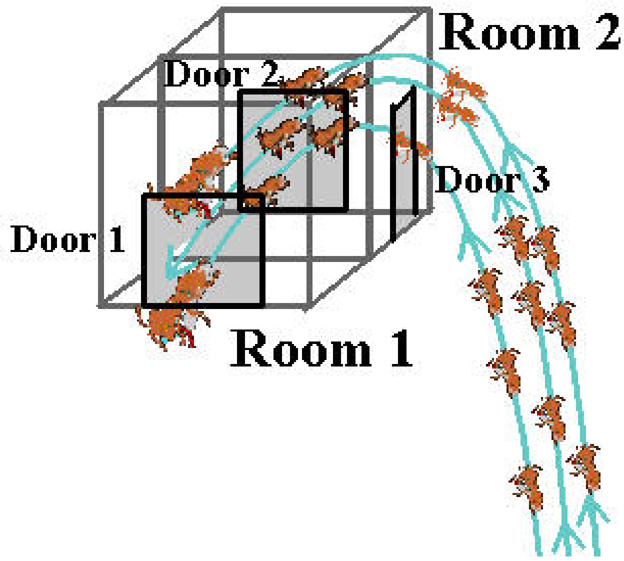
$$\Phi_S = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$




Room 2
Door 2
Door 3
Door 1
Room 1

Ants enter Room 2 through Door 3 at 4 ants/sec
Ants are passing through Door 2 into Room 1 at 2 ants/sec
Ants are leaving through Door 1 at 1 ant/sec

Therefore, the ant flux is -3 ants/sec



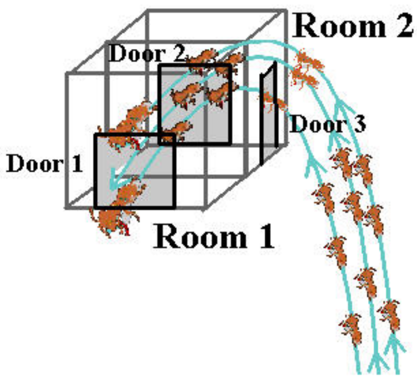
Camera in Room 1 shows an ant flux of -1

$$\text{Flux out of Room 1} = -2 + 1 = -1$$

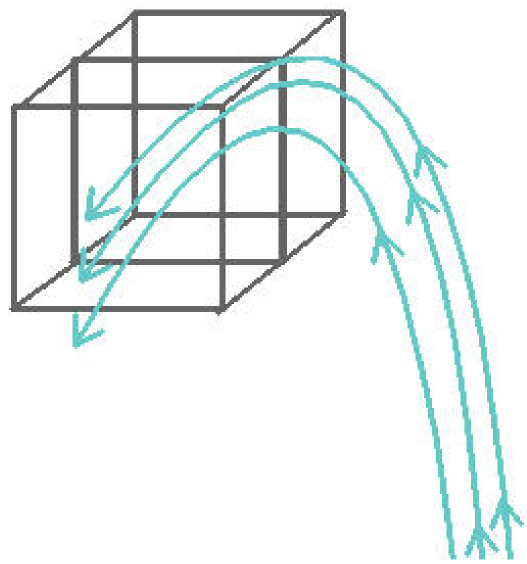
Camera in Room 2 shows an ant flux of -2

$$\text{Flux out of Room 2} = -4 + 2 = -2$$

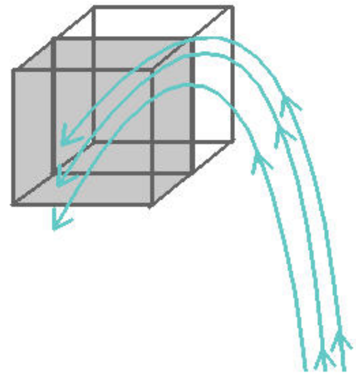
$$\text{Flux (Room 1)} + \text{Flux (Room 2)} = (-2 + 1) + (-4 + 2) = -3$$



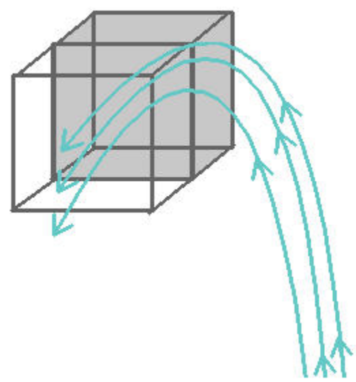
Back to general vector fields



Each compartment has six faces.
first compartment: $S_{11},\ S_{12},\ S_{13},\ S_{14},\ S_{15}$ and S_{16}



second compartment: $S_{21},\ S_{22},\ S_{23},\ S_{24},\ S_{25}$ and S_{26}.



$$S_{16} = S_{26}$$

The total flux out of the first compartment is:

$$\Phi_{S_{11}} + \Phi_{S_{12}} + \Phi_{S_{13}} + \Phi_{S_{14}} + \Phi_{S_{15}} + \Phi_{S_{16}} = \sum_{j=1}^{6} \Phi_{S_{1j}}$$

and the total flux out of the second compartment is:

$$\Phi_{S_{21}} + \Phi_{S_{22}} + \Phi_{S_{23}} + \Phi_{S_{24}} + \Phi_{S_{25}} + \Phi_{S_{26}} = \sum_{j=1}^{6} \Phi_{S_{2j}}$$

$$\vec{\mathbf{n}}_{26} = -\vec{\mathbf{n}}_{16}$$

$$\int_{S_{16}} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}_{16}\, dS = \int_{S_{16}} \vec{\mathbf{F}} \bullet (-\vec{\mathbf{n}}_{26})\, dS = -\int_{S_{26}} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}_{26}\, ds$$

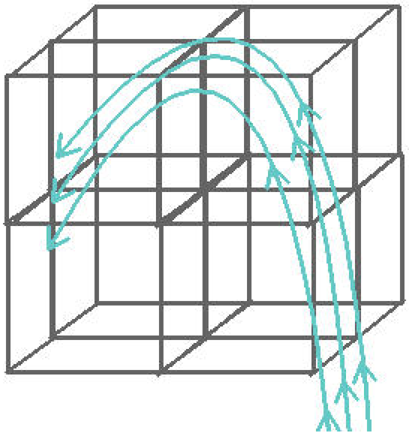
$$\Phi_{S_{16}} = -\Phi_{S_{26}}$$

$$
\begin{aligned}
\sum_{j=1}^{6}\Phi_{S_{1j}} + \sum_{j=1}^{6}\Phi_{S_{2j}} \\
&= \left(\sum_{j=1}^{5}\Phi_{S_{1j}} + \Phi_{S_{16}}\right) + \left(\sum_{j=1}^{5}\Phi_{S_{2j}} + \Phi_{S_{26}}\right) \\
&= \left(\sum_{j=1}^{5}\Phi_{S_{1j}} - \Phi_{S_{26}}\right) + \left(\sum_{j=1}^{5}\Phi_{S_{2j}} + \Phi_{S_{26}}\right) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{5}\Phi_{S_{ij}}
\end{aligned}
$$

$$\sum_{i=1}^{2}\sum_{j=1}^{6}\Phi_{S_{ij}} = \sum_{\text{exterior faces}}\sum \Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

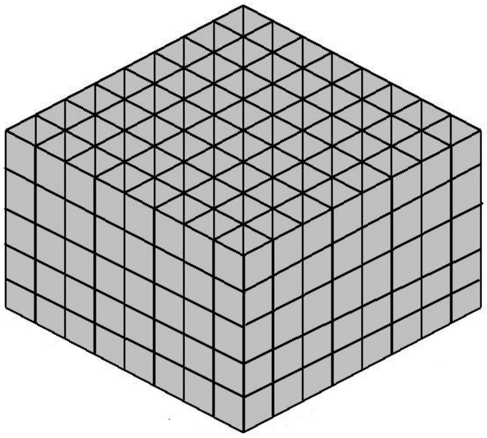
where S is the entire surface surrounding the two combined compartments.

Generalize to more interior compartments.



$$\sum_{i=1}^{8}\sum_{j=1}^{6}\Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

n compartments.

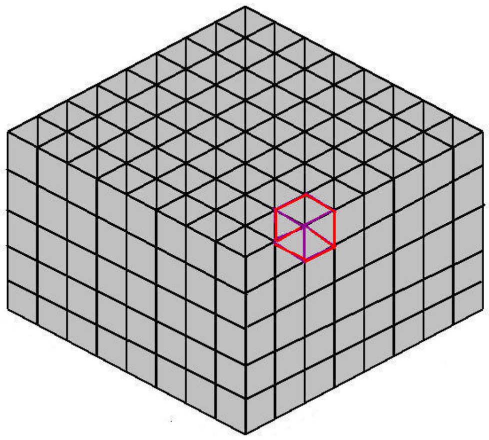


$$\sum_{i=1}^{n} \sum_{j=1}^{6} \Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}} \, dS$$

$$\nabla \bullet \vec{\mathbf{F}} = \lim_{vol(V) \to 0} \frac{\Phi_S}{\text{vol}(V)}$$

So, in one small compartment, $\nabla \bullet \vec{\mathbf{F}} \approx \frac{\Phi_S}{\text{vol}(V)}$

$$\Phi_S \approx (\nabla \bullet \vec{\mathbf{F}})\text{vol}(V)$$



$$\sum_{j=1}^{6} \Phi_{S_{ij}} \approx (\nabla \bullet \vec{\mathbf{F}})\mathrm{vol}(V_i)$$

where V_i is the interior of the i^{th} compartment.

$$\sum_{j=1}^{6} \Phi_{S_{ij}} \approx (\nabla \bullet \vec{\mathbf{F}}) \cdot \text{vol}(V_i)$$

where V_i is the interior of the i^{th} compartment.

$$\sum_{i=1}^{n} \nabla \bullet \vec{\mathbf{F}} \, \text{vol}(V_i) \approx \sum_{i=1}^{n} \sum_{j=1}^{6} \Phi_{S_{ij}} = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}} \, dS$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\nabla\bullet\vec{\mathbf{F}}\,\mathrm{vol}(V_i)=\iint_S\vec{\mathbf{F}}\bullet\vec{\mathbf{n}}\,dS$$

$$\iiint_V\nabla\bullet\vec{\mathbf{F}}\,dV=\iint_S\vec{\mathbf{F}}\bullet\vec{\mathbf{n}}\,dS$$

Divergence Theorem.

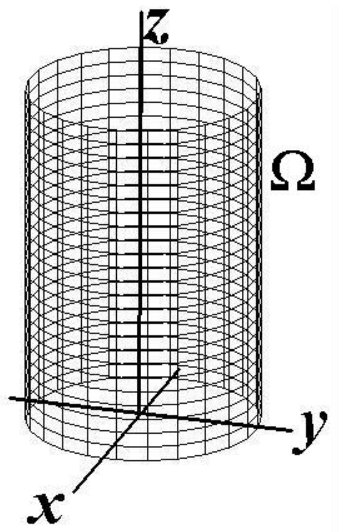
Divergence Theorem - Conditions

1. $\nabla \bullet \vec{\mathbf{F}}$ must exist at all points in the interior of the solid

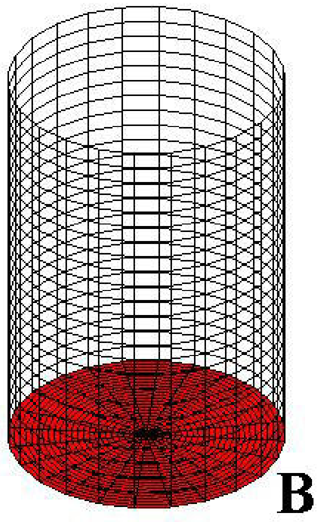
2. S must be a *closed* surface.

$$\iiint_V \nabla \bullet \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

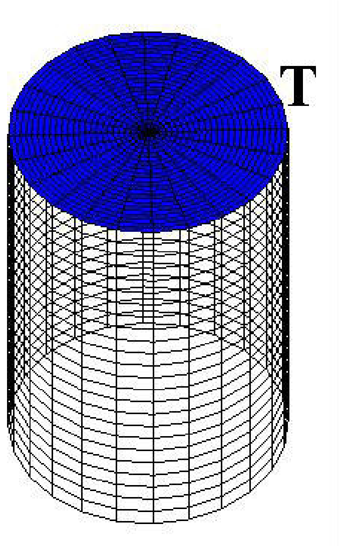
Example: Let Ω be the surface forming the vertical side of a cylinder of radius 1 around the z axis, for $0 \leq z \leq 3$



Let B be the surface that forms the base of this cylinder. B is a disk of radius 1 in the xy plane.

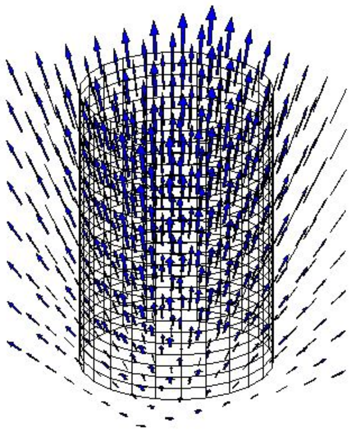


Let T be the surface that forms the top of this cylinder. T is a disk of radius 1 that is 3 units above the xy plane.



Let S be the closed surface around the cylinder
Let $\vec{\mathbf{F}} = \langle x,\ y,\ z+1 \rangle$.

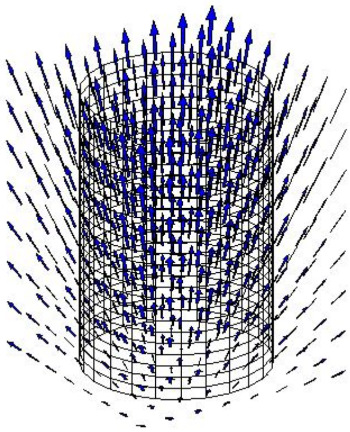
Calculate the surface integral: $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$



Calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ by adding up the flux through all three surfaces that form S

$$\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}} \, dS = 6\pi - \pi + 4\pi = 9\pi$$

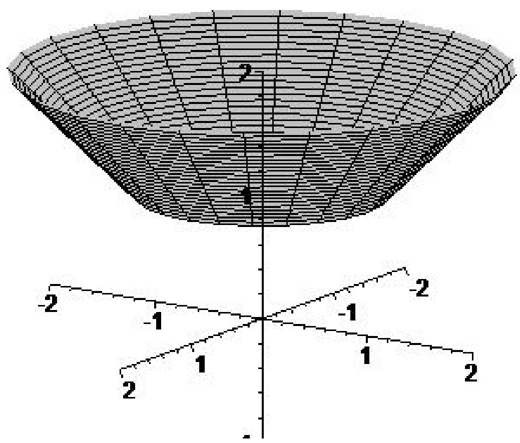


If $\vec{\mathbf{F}} = \langle x,\ y,\ z+1 \rangle$ then:

$$\operatorname{div} \vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z+1) = 3$$

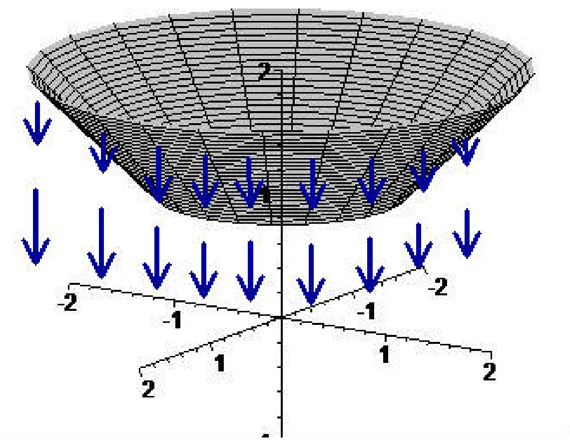
$$\begin{aligned}\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iiint_V \nabla \bullet \vec{\mathbf{F}}\, dV \\ &= \iiint_V 3\, dV \\ &= 3 \cdot \text{Vol}(V) = 3 \cdot 3\pi = 9\pi\end{aligned}$$

Let Ω be the portion of the cone $z = \sqrt{x^2 + y^2}$ for $1 \leq z \leq 2$

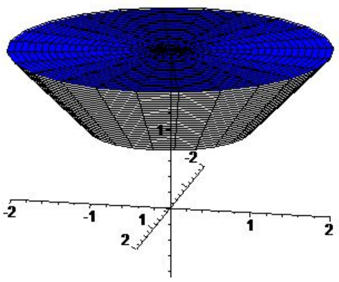


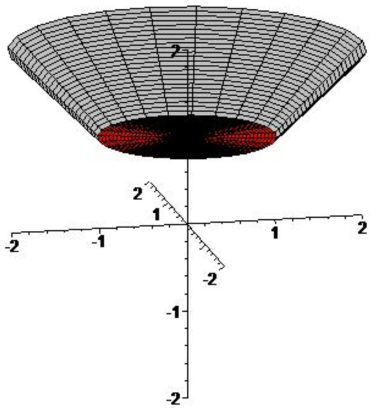
$$\vec{\mathbf{F}} = \left(2z^2 - 8\right)\vec{\mathbf{k}}$$

$$\text{Calculate } \iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$



Add in the top T and the bottom B. Call the closed surface S





$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$$

Along the top, $z = 2$ so $\vec{\mathbf{F}} = \left(2z^2 - 8\right)\vec{\mathbf{k}} = 0\vec{\mathbf{k}}$

$$\iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = 0$$

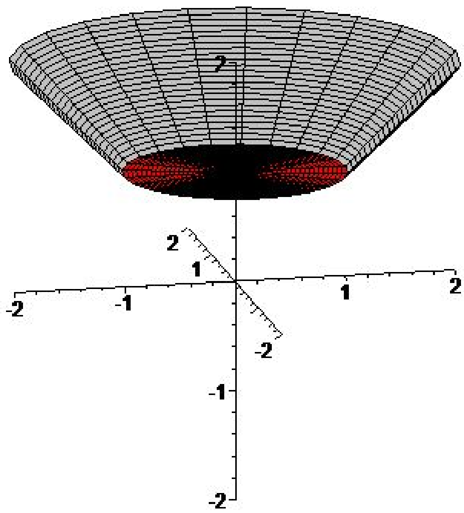
$$
\begin{aligned}
\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \\
&= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + 0 + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS
\end{aligned}
$$

Along the bottom, $z = 1$ so $\vec{\mathbf{F}} = \left(2z^2 - 8\right)\vec{\mathbf{k}} = -6\vec{\mathbf{k}}$

The outward normal on the bottom is $\vec{\mathbf{n}} = -\vec{\mathbf{k}}$

$$\vec{\mathbf{F}} \bullet \vec{\mathbf{n}} = 6$$

$$\iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_B 6\, dS = 6 \cdot \text{Area}(B) = 6\pi$$



$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \\
&= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + 0 + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \\
&= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + 0 + 6\pi
\end{aligned}$$

$$\begin{aligned}\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \\ &= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + 0 + \iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS \\ &= \iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS + 0 + 6\pi\end{aligned}$$

$$\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS - 6\pi$$

$$\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{S} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS - 6\pi$$

$$= \iiint_{V} \nabla \bullet \vec{\mathbf{F}}\, dV - 6\pi$$

If $\vec{\mathbf{F}} = \left(2z^2 - 8\right) \vec{\mathbf{k}}$ then $\nabla \bullet \vec{\mathbf{F}} = 4z$

$$
\begin{aligned}
\iint_{\Omega} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iint_{S} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS - 6\pi \\
&= \iiint_{V} \nabla \bullet \vec{\mathbf{F}}\, dV - 6\pi \\
&= \iiint_{V} 4z\, dV - 6\pi \\
&= \int_{1}^{2} \int_{0}^{2\pi} \int_{0}^{z} 4z\, r\, dr\, d\theta\, dz - 6\pi \\
&= 15\pi - 6\pi \\
&= 9\pi
\end{aligned}
$$