∇ Notation and Identities

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$$\nabla \bullet \vec{\mathbf{F}}$$

$$\nabla\phi$$

$$\nabla \times \vec{\mathbf{F}}$$

$$\nabla^2\phi$$

$$f = f(x)$$

$$f'(x) = \frac{d}{dx} f(x)$$

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Gradient Operator

If $\phi = \phi(x, y, z)$ then the gradient of ϕ is:

$$\left\langle \frac{\partial \phi}{\partial x}, \ \frac{\partial \phi}{\partial y}, \ \frac{\partial \phi}{\partial z} \right\rangle$$

Or, equivalently:

$$\frac{\partial \phi}{\partial x}\,\vec{\mathbf{i}} + \frac{\partial \phi}{\partial y}\,\vec{\mathbf{j}} + \frac{\partial \phi}{\partial z}\,\vec{\mathbf{k}}$$

Example:

$$\phi(x, y, z) = x^2y + z^3$$

The gradient is the vector:

$$\left\langle 2xy,\ x^2,\ 3z^2 \right\rangle$$

$$\phi = \phi(x, y, z)$$

$$\left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y},\ \frac{\partial \phi}{\partial z} \right\rangle = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \phi$$

Abbreviation:

$$\nabla \phi$$

$$\nabla = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle$$

$$\nabla = \frac{\partial}{\partial x}\,\vec{\mathbf{i}} + \frac{\partial}{\partial y}\,\vec{\mathbf{j}} + \frac{\partial}{\partial z}\,\vec{\mathbf{k}}$$

$$\nabla = \frac{\partial}{\partial x}\vec{\mathbf{i}} + \frac{\partial}{\partial y}\vec{\mathbf{j}} + \frac{\partial}{\partial z}\vec{\mathbf{k}}$$

Warning, could be misinterpreted as

$$\vec{\mathbf{0}} + \vec{\mathbf{0}} + \vec{\mathbf{0}}$$

$$\nabla = \vec{\mathbf{i}}\,\frac{\partial}{\partial x} + \vec{\mathbf{j}}\,\frac{\partial}{\partial y} + \vec{\mathbf{k}}\,\frac{\partial}{\partial z}$$

Derivative Properties:

If $f = f(x)$, $g = g(x)$ and both a and b are constants then:

$$\frac{d}{dx}(af + bg) = a\frac{df}{dx} + b\frac{dg}{dx}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Gradient Properties:

If $f = f(x, y, z)$ and $g = g(x, y, z)$ then:

$$\nabla(af + bg) = a\nabla f + b\nabla g$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$
\begin{aligned}
\nabla(af+bg) &= \left\langle \frac{\partial}{\partial x}(af+bg),\ \frac{\partial}{\partial y}(af+bg),\ \frac{\partial}{\partial z}(af+bg) \right\rangle \\
&= \left\langle a\frac{\partial f}{\partial x}+b\frac{\partial g}{\partial x},\ a\frac{\partial f}{\partial y}+b\frac{\partial g}{\partial y},\ a\frac{\partial f}{\partial z}+b\frac{\partial g}{\partial z} \right\rangle \\
&= a\left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right\rangle + b\left\langle \frac{\partial g}{\partial x},\ \frac{\partial g}{\partial y},\ \frac{\partial g}{\partial z} \right\rangle \\
&= a\nabla f + b\nabla g
\end{aligned}
$$

If $F = \langle F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)\rangle$ then the divergence of $\vec{\mathbf{F}}$ is:

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example: If $\vec{\mathbf{F}} = \langle xyz^3,\ y^2 z,\ xz^2\rangle$ then:

$$\operatorname{div} \vec{\mathbf{F}} = yz^3 + 2yz + 2xz$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial z} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \langle F_1,\ F_2,\ F_3 \rangle$$

$$\operatorname{div} \vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}}$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial z} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \langle F_1,\ F_2,\ F_3 \rangle$$

$$\operatorname{div} \vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}}$$

Also can be written as:

$$\left(\vec{\mathbf{i}} \frac{\partial}{\partial x} + \vec{\mathbf{j}} \frac{\partial}{\partial y} + \vec{\mathbf{k}} \frac{\partial}{\partial z} \right) \bullet \left(F_1 \vec{\mathbf{i}} + F_2 \vec{\mathbf{j}} + F_3 \vec{\mathbf{k}} \right)$$

Derivative Properties:

If $f = f(x)$, $g = g(x)$ and both a and b are constants then:

$$\frac{d}{dx}(af + bg) = a\frac{df}{dx} + b\frac{dg}{dx}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Divergence Properties:

If $\phi = \phi(x, y, z)$ is a scalar-valued function and $\vec{\mathbf{F}} = \vec{\mathbf{F}}(x, y, z)$ and $\vec{\mathbf{G}} = \vec{\mathbf{G}}(x, y, z)$ are both vector fields then:

$$\nabla \bullet (a\vec{\mathbf{F}} + b\vec{\mathbf{G}}) = a\nabla \bullet \vec{\mathbf{F}} + b\nabla \bullet \vec{\mathbf{G}}$$

$$\nabla \bullet (\phi\vec{\mathbf{F}}) = \phi\nabla \bullet \vec{\mathbf{F}} + \nabla\phi \bullet \vec{\mathbf{F}}$$

If $\vec{\mathbf{F}} = \langle F_1,\ F_2,\ F_3 \rangle$ then $\phi\vec{\mathbf{F}} = \langle \phi F_1,\ \phi F_2,\ \phi F_3 \rangle$

$$\nabla \bullet (\phi\vec{\mathbf{F}}) = \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3)$$

Expand, using the Product Rule for each term:

$$\phi\frac{\partial F_1}{\partial x} + \frac{\partial \phi}{\partial x}F_1 + \phi\frac{\partial F_2}{\partial y} + \frac{\partial \phi}{\partial y}F_2 + \phi\frac{\partial F_3}{\partial z} + \frac{\partial \phi}{\partial z}F_3$$

Rearrange terms:

$$\phi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + \frac{\partial \phi}{\partial x}F_1 + \frac{\partial \phi}{\partial y}F_2 + \frac{\partial \phi}{\partial x}F_3$$

$$\phi\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)+\frac{\partial \phi}{\partial x}F_1+\frac{\partial \phi}{\partial y}F_2+\frac{\partial \phi}{\partial x}F_3$$

This is the same as:

$$\phi\nabla \bullet \vec{\mathbf{F}}+\nabla\phi\bullet\vec{\mathbf{F}}$$

We can combine operations:

$$\operatorname{div}(\operatorname{grad}\phi) = \nabla \bullet \nabla\phi$$

$$\nabla \bullet \nabla\phi = \left\langle \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z} \right\rangle \bullet \left\langle \frac{\partial\phi}{\partial x}, \ \frac{\partial\phi}{\partial y}, \ \frac{\partial\phi}{\partial x} \right\rangle$$
$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Abbreviation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Example: Let $\phi = x^2 y^2 z^3$

$$\nabla^2 \phi = 2y^2 z^3 + 2x^2 z^3 + 6x^2 y^2 z$$

The Wave Equation: Let $u = u(x, y, z, t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

More compactly,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Schrödinger's Equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) + V\Psi$$

More Compactly:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \\ F_1 & F_2 \end{vmatrix} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$\mathbf{curl}\,\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$
\begin{aligned}
\nabla \times \vec{\mathbf{F}} &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \vec{\mathbf{k}}
\end{aligned}
$$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \vec{\mathbf{k}}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \vec{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{\mathbf{k}}$$

Example: Let $\vec{\mathbf{F}} = \left\langle x+z,\ x+y,\ z^2 \right\rangle$

Calculate $\nabla \times \vec{\mathbf{F}}$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z & x+y & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & z^2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x+z & z^2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x+z & x+y \end{vmatrix}$$

$$= (0-0)\vec{\mathbf{i}} - (0-1)\vec{\mathbf{j}} + (1-0)\vec{\mathbf{k}}$$

$$= \vec{\mathbf{j}} + \vec{\mathbf{k}}$$

Example: Let $\vec{\mathbf{F}} = \left\langle 2xy,\ x^2,\ 3z^2 \right\rangle$.

Calculate $\nabla \times \vec{\mathbf{F}}$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 & 3z^2 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3z^2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xy & 3z^2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy & x^2 \end{vmatrix} \vec{\mathbf{k}}$$

$$= (0 - 0)\,\vec{\mathbf{i}} - (0 - 0)\,\vec{\mathbf{j}} + (2x - 2x)\,\vec{\mathbf{k}}$$

$$= \vec{\mathbf{0}}$$

$$\phi = x^2 y + z^3$$

$$\nabla \phi = \left\langle 2xy,\ x^2,\ 3z^2 \right\rangle$$

$$\nabla \times \nabla \phi = \vec{\mathbf{0}}$$

$$\vec{\mathbf{F}} = \left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y},\ \frac{\partial \phi}{\partial z} \right\rangle$$

$\nabla \times \vec{\mathbf{F}}$ comes out to:

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{\mathbf{k}}$$

In this case, it equals:

$$\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)\vec{\mathbf{i}} - \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right)\vec{\mathbf{i}} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)\vec{\mathbf{k}}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

$$\nabla \times \nabla\phi = 0\vec{\mathbf{i}} + 0\vec{\mathbf{j}} + 0\vec{\mathbf{k}}$$