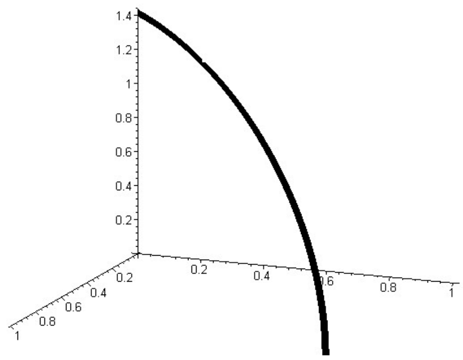
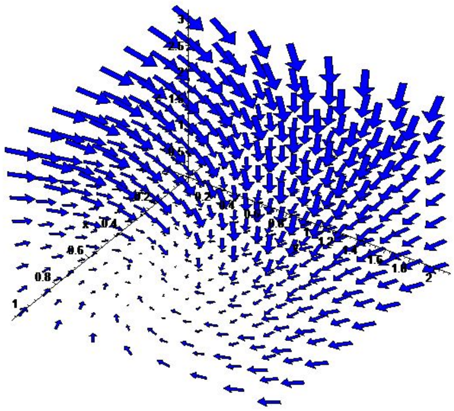
Line Integrals - More Review

Dr. Elliott Jacobs

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



$$\vec{\mathbf{F}} = \langle y - x,\ z - y,\ x - z \rangle$$



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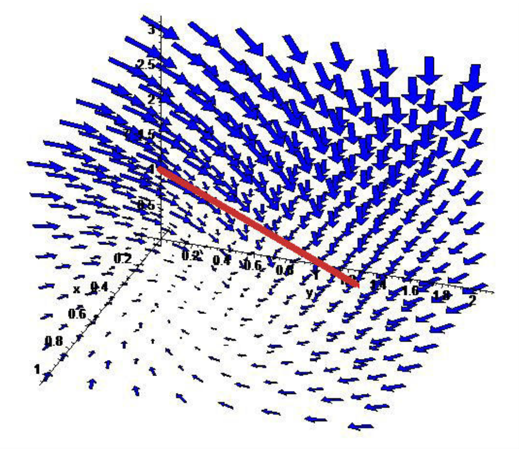

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Let $\vec{\mathbf{F}} = \langle y - x,\ z - y,\ x - z \rangle$. Integrate this vector field over the straight line segment L connecting $(0, 0, 1)$ to $(1, 2, 2)$.
The straight line L can be described by the equation:

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$$

$$\vec{\mathbf{r}} = \langle 0, 0, 1 \rangle + t\langle 1, 2, 1 \rangle = \langle t,\ 2t,\ 1 + t \rangle \qquad \text{for} \quad 0 \le t \le 1$$

At any point on this line, $\vec{\mathbf{F}}$ is given by:

$$\vec{\mathbf{F}} = \langle y - x,\ z - y,\ x - z \rangle = \langle t,\ 1 - t,\ -1 \rangle$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle t,\ 2t,\ 1+t \rangle$$

The vector $d\vec{\mathbf{r}}$ is given by:

$$d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\,dt = \langle 1, 2, 1 \rangle\,dt$$

$$\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \langle t,\ 1-t,\ -1 \rangle \bullet \langle 1, 2, 1 \rangle\,dt = (t+2(1-t)-1)\,dt = (1-t)\,dt$$

The line integral is therefore equal to:

$$\int_L \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^1 (1-t)\,dt = \frac{1}{2}$$

Alternate Solution:

$$d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\,dt = \left\langle \frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt} \right\rangle dt \ = \langle dx,\ dy,\ dz \rangle$$

If $\vec{\mathbf{F}} = \langle F_1,\ F_2,\ F_3 \rangle$ then $\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = F_1\,dx + F_2\,dy + F_3\,dz$ and so the line integral can be written as:

$$\int_L \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_L F_1\,dx + F_2\,dy + F_3\,dz$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle t,\ 2t,\ 1+t\rangle$$

$$\int_L \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_L (y-x)\,dx + (z-y)\,dy + (x-z)\,dz$$

Since $x = t$, $y = 2t$ and $z = 1 + t$, we have:

$$y = 2x \qquad z = 1 + x$$

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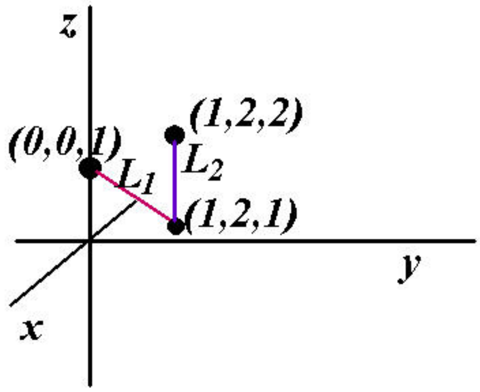
$$\int_L \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_L (y - x)\,dx + (z - y)\,dy + (x - z)\,dz$$

$$= \int_0^1 (2x - x)\,dx + ((1 + x) - 2x)\,2\,dx + (x - (1 + x))\,dx$$

$$= \int_0^1 (1 - x)\,dx = \frac{1}{2}$$

Change the path.
Let L_1 denote the straight line segment from $(0, 0, 1)$ to $(1, 2, 1)$ and L_2 denote the straight line segment from $(1, 2, 1)$ to $(1, 2, 2)$ and let L_3 be the combined path along L_1 followed by L_2.



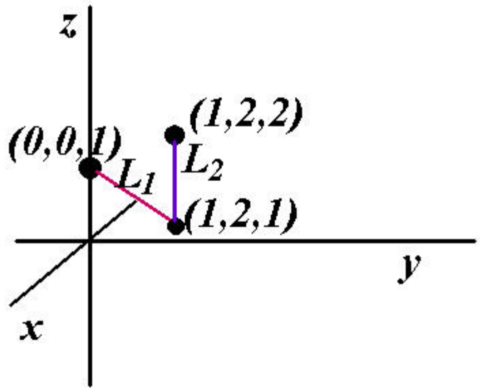
The work done along the combined path L_3 is the sum of the work done along L_1 and the work done along L_2.

$$\int_{L_3} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{L_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{L_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

Along L_1, $y = 2x$ and $z = 1$ so $dy = 2\,dx$ and $dz = 0$.

$$(y-x)\,dx+(z-y)\,dy = (2x-x)\,dx+(1-2x)\cdot 2\,dx = (2-3x)\,dx$$

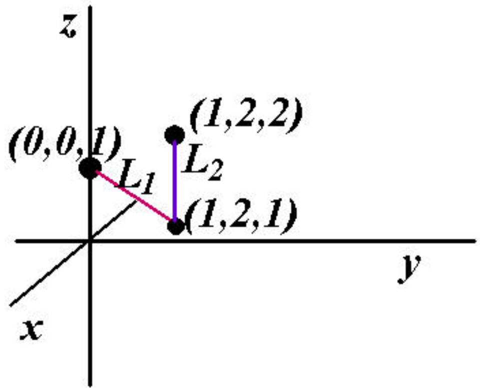
$$\int_{L_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{L_1} (y-x)\,dx + (z-y)\,dy = \int_0^1 (2-3x)\,dx = \frac{1}{2}$$



$$\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = F_1\,dx + F_2\,dy + F_3\,dz$$

Along L_2, $x = 1$ and $y = 2$ at every point so $dx = dy = 0$.

$$\int_{L_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{L_2} (x - z)\,dz = \int_1^2 (1 - z)\,dz = -\frac{1}{2}$$



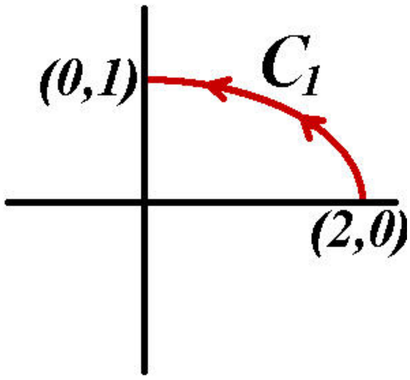
$$\int_{L_3} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{L_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{L_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \frac{1}{2} - \frac{1}{2} = 0$$

For $\vec{\mathbf{F}} = \langle y - x,\ x - y,\ x - z \rangle$, changing the path between the initial and final points changed the value of the line integral.

The integral is said to be *path dependent* and $\vec{\mathbf{F}}$ is called a *nonconservative vector field.*

Example:

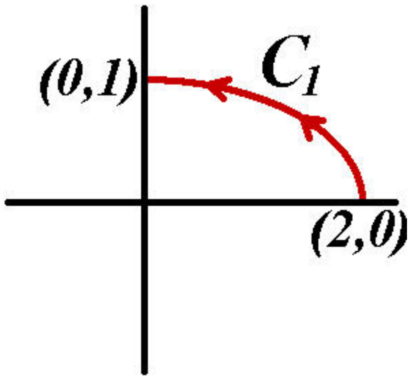
Let C_1 be the path from $(2, 0)$ to $(0, 1)$ along the upper portion of the ellipse $\frac{x^2}{4} + y^2 = 1$.



$$\vec{\mathbf{F}} = \langle -x + y,\ x - 2y \rangle$$

Calculate the line integral:

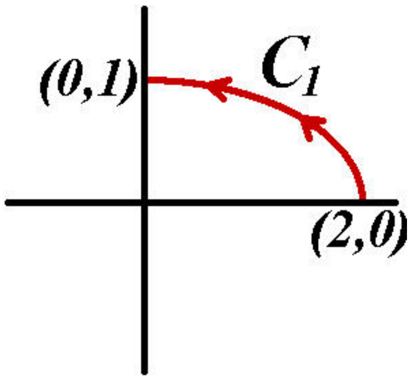
$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} F_1\, dx + F_2\, dy + F_3\, dz$$

If the path is in the xy plane then $dz = 0$

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} F_1\, dx + F_2\, dy$$



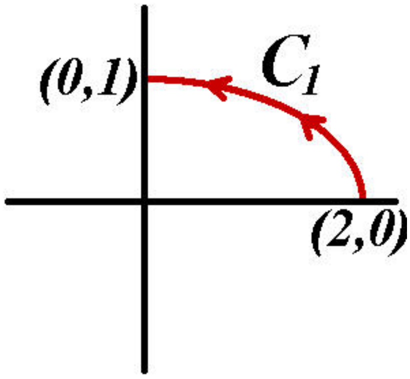
On this elliptical path, $\frac{x^2}{4} + y^2 = 1$, we can solve for y

$$y = \frac{1}{2}\sqrt{4 - x^2} \qquad\qquad dy = \frac{-x}{2\sqrt{4 - x^2}}\, dx$$

$$\begin{aligned}
\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= (-x + y)\, dx + (x - 2y)\, dy \\
&= (-x + \frac{1}{2}\sqrt{4 - x^2})\, dx + (x - \sqrt{4 - x^2}) \left(\frac{-x}{2\sqrt{4 - x^2}} \right) dx \\
&= -\frac{x}{2}\, dx + \frac{2 - x^2}{\sqrt{4 - x^2}}\, dx
\end{aligned}$$

We must start at $x = 2$ and end up at $x = 0$ if we want to integrate in the direction specified.

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = -\int_2^0 \frac{x}{2}\,dx + \int_2^0 \frac{2-x^2}{\sqrt{4-x^2}}\,dx = 1 - \int_0^2 \frac{2-x^2}{\sqrt{4-x^2}}\,dx$$



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Use trigonometric substitution:

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 1$$

Alternate method: use the equations $\quad x = 2\cos t \quad y = \sin t$

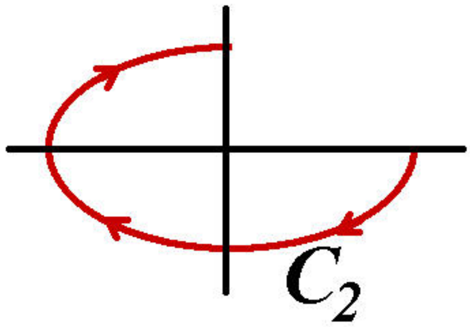
$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} (-x + y)\, dx + (x - 2y)\, dy$$

$$= \int_0^{\pi/2} (-2\cos t + \sin t)(-2\sin t\, dt) + (2\cos t - 2\sin t)(\cos t\, dt)$$

$$= \int_0^{\pi/2} (2\sin t \cos t + 2(\cos^2 t - \sin^2 t))\, dt$$

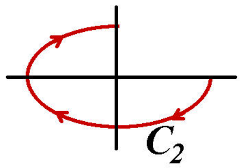
$$= \int_0^{\pi/2} (\sin 2t + 2\cos 2t)\, dt = 1$$

Let C_2 be the elliptical path along the other three quarters of the ellipse.



To traverse path C_2, we simply vary t from 2π to $\frac{\pi}{2}$.

$$\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{2\pi}^{\pi/2} (\sin 2t + 2\cos 2t)\, dt = 1$$



$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

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Does this mean that the integral of $\vec{\mathbf{F}}$ from $(2,0)$ to $(0,1)$ is path independent? What if there is some other path, call it C_3 from $(2,0)$ to $(0,1)$ where the answer for $\int_{C_3} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ comes out differently?