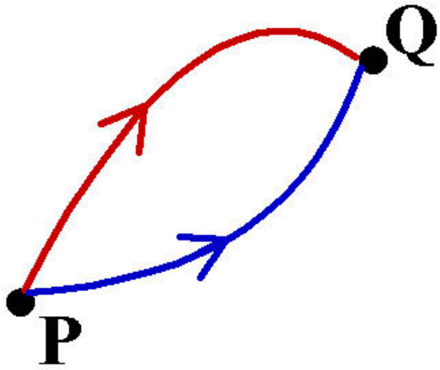
Conservative Vector Fields

Dr. Elliott Jacobs

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



Conservative Vector Fields

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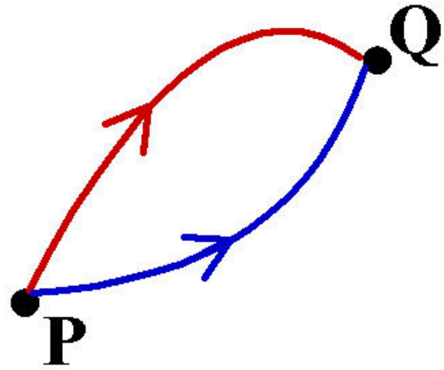
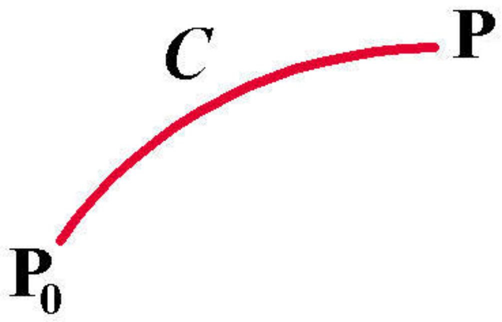
$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



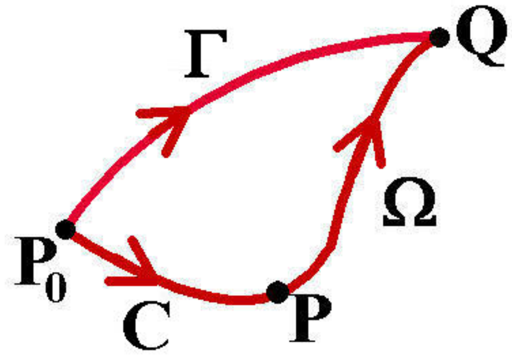
The Potential Function

$$\phi(\mathbf{P}) = \int_{\mathbf{P_0}}^{\mathbf{P}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



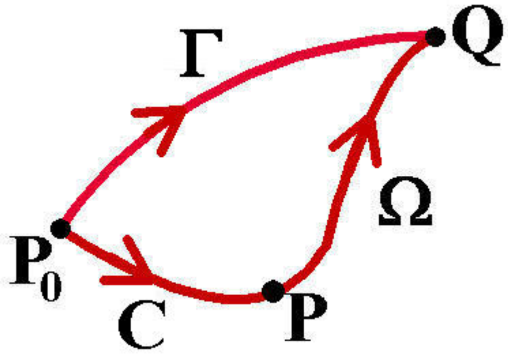
If **P** and **Q** are two different points then:

$$\int_{\mathbf{C}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{\mathbf{\Omega}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{\mathbf{\Gamma}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



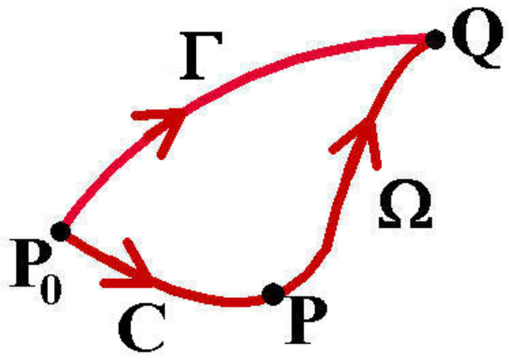
If **P** and **Q** are two different points then:

$$\int_{\mathbf{P_0}}^{\mathbf{P}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{\mathbf{P_0}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



$$\int_{\mathbf{P_0}}^{\mathbf{P}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{\mathbf{P_0}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

$$\phi(\mathbf{P}) + \int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q})$$

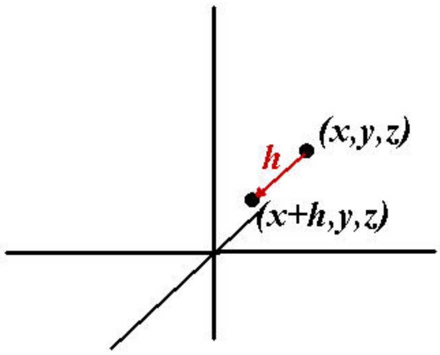


$$\int_{\mathbf{P_0}}^{\mathbf{P}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{\mathbf{P_0}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

$$\phi(\mathbf{P}) + \int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q})$$

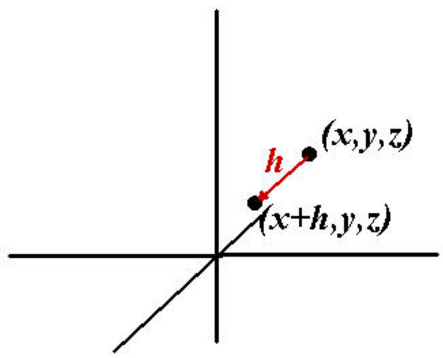
$$\int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q}) - \phi(\mathbf{P})$$

$$\int_{(x,y,z)}^{(x+h,y,z)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(x+h,y,z) - \phi(x,y,z)$$



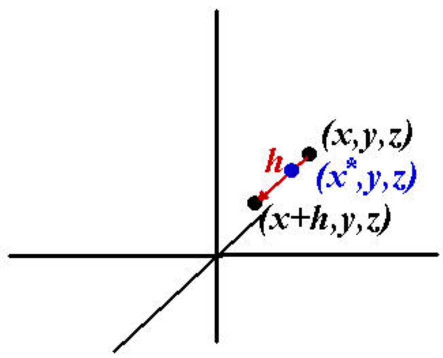
$$\int_{(x,y,z)}^{(x+h,y,z)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{(x,y,z)}^{(x+h,y,z)} F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \int_{(x,y,z)}^{(x+h,y,z)} F_1\, dx$$



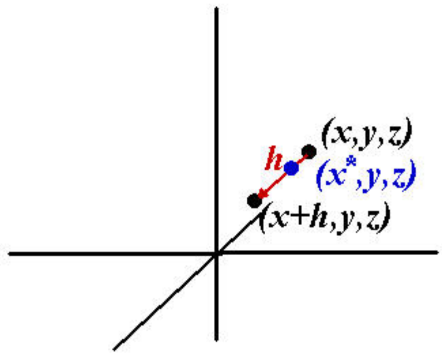
$$\int_{(x,y,z)}^{(x+h,y,z)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{(x,y,z)}^{(x+h,y,z)} F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \int_{(x,y,z)}^{(x+h,y,z)} F_1\, dx = F_1(x^*, y, z)h$$



$$\int_{(x,y,z)}^{(x+h,y,z)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(x+h,y,z) - \phi(x,y,z)$$

$$F_1(x^*,y,z)h = \phi(x+h,y,z) - \phi(x,y,z)$$

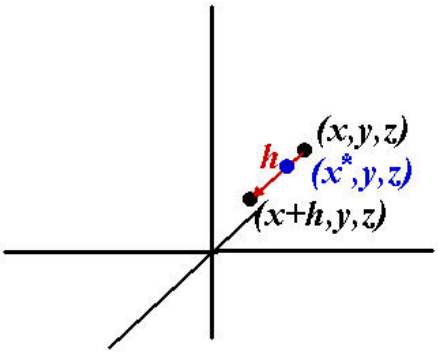


$$F_1(x^*, y, z)h) = \phi(x + h, y, z) - \phi(x, y, z)$$

$$F_1(x^*, y, z) = \frac{\phi(x + y, y, z) - \phi(x, y, z)}{h}$$

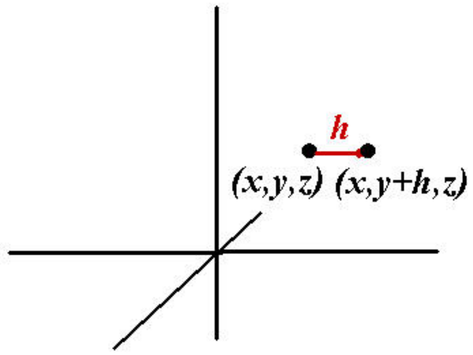
Take the limit as $h \to 0$

$$F_1(x, y, z) = \frac{\partial \phi}{\partial x}$$



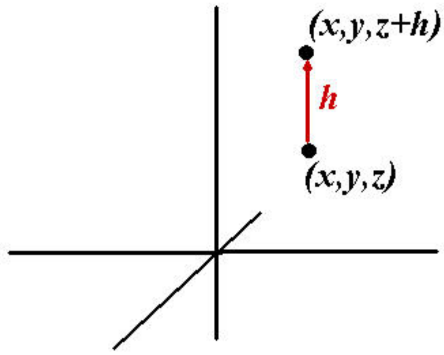
$$\int_{(x,y,z)}^{(x,y+h,z)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(x, y+h, z) - \phi(x, y, z)$$

$$F_2(x, y, z) = \frac{\partial \phi}{\partial y}$$



$$\int_{(x,y,z)}^{(x,y,z+h)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(x,y,z+h) - \phi(x,y,z)$$

$$F_3(x,y,z) = \frac{\partial \phi}{\partial z}$$



$$\vec{\mathbf{F}} = \langle F_1,\ F_2,\ F_3 \rangle = \left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y},\ \frac{\partial \phi}{\partial z} \right\rangle$$

Theorem
If $\vec{\mathbf{F}}$ is a conservative vector field then there is a scalar-valued function $\phi(x, y, z)$ such that:

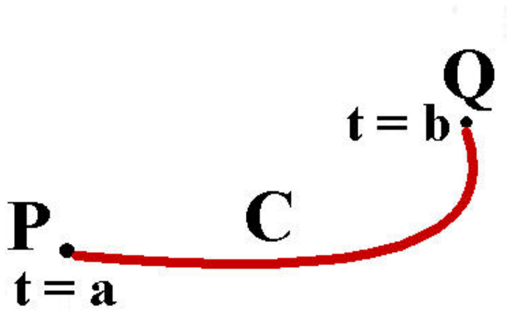
$$\vec{\mathbf{F}} = \nabla\phi$$

Suppose $\vec{\mathbf{F}} = \nabla\phi$ for some scalar-valued function $\phi = \phi(x, y, z)$. Can we conclude that $\vec{\mathbf{F}}$ *must* be a conservative vector field?

Suppose we have some curve connecting **P** to **Q**

$$x = x(t) \qquad y = y(t) \qquad z = z(t)$$

At $t = a$, the curve begins at **P**. At $t = b$ the curve ends at **Q**



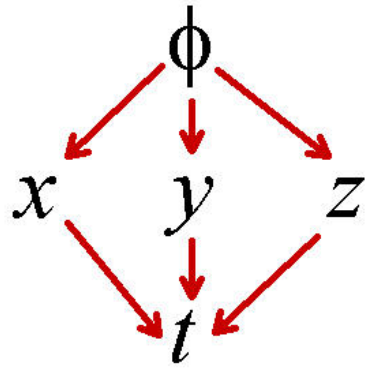
$$\vec{\mathbf{F}} = \nabla \phi$$

$$\langle F_1,\ F_2,\ F_3 \rangle = \left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y},\ \frac{\partial \phi}{\partial z} \right\rangle$$

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \int_a^b \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt \\
&= \int_a^b \left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y},\ \frac{\partial \phi}{\partial z} \right\rangle \bullet \left\langle \frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt} \right\rangle dt \\
&= \int_a^b \left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \right) dt
\end{aligned}$$

$$\phi(x, y, z) = \phi(x(t), y(t), z(t))$$

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$



$$
\begin{aligned}
\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \int_a^b \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\,dt \\
&= \int_a^b \left\langle \frac{\partial \phi}{\partial x},\ \frac{\partial \phi}{\partial y},\ \frac{\partial \phi}{\partial z} \right\rangle \bullet \left\langle \frac{dx}{dt}\ \frac{dy}{dt},\ \frac{dz}{dt} \right\rangle dt \\
&= \int_a^b \left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d\phi}{dt}\,dt \\
&= \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a)) \\
&= \phi(\mathbf{Q}) - \phi(\mathbf{P})
\end{aligned}
$$

Theorem:

$\vec{\mathbf{F}}$ is a conservative vector field if and only if there is a scalar-valued function ϕ such that $\vec{\mathbf{F}} = \nabla\phi$

If $\vec{\mathbf{F}}$ is a conservative vector field then:

$$\int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q}) - \phi(\mathbf{P})$$

$$\int_{\mathbf{P}}^{\mathbf{Q}} \nabla\phi \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q}) - \phi(\mathbf{P})$$

Fundamental Theorem for Line Integrals

In Calculus I:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

And now:

$$\int_{\mathbf{P}}^{\mathbf{Q}} \nabla\phi \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q}) - \phi(\mathbf{P})$$