The Potential Function

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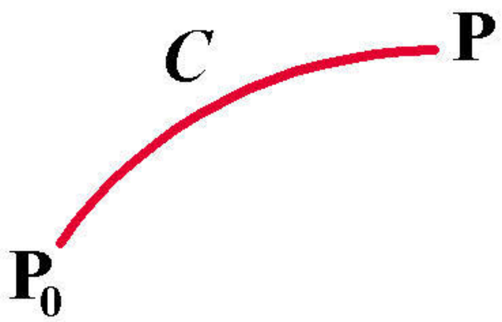
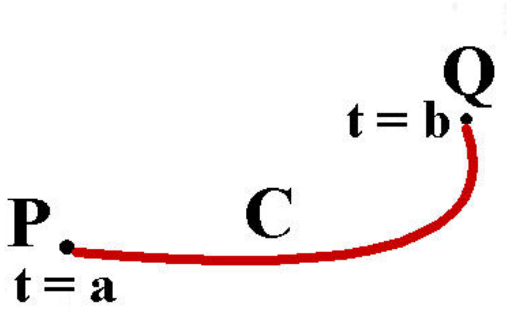
$$\phi(\mathbf{P}) = \int_{\mathbf{P_0}}^{\mathbf{P}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



If $\vec{\mathbf{F}}$ is a conservative vector field then:

$$\int_{\mathbf{P}}^{\mathbf{Q}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q}) - \phi(\mathbf{P})$$



Theorem
If $\vec{\mathbf{F}}$ is a conservative vector field then there is a scalar-valued function $\phi(x, y, z)$ such that:

$$\vec{\mathbf{F}} = \nabla\phi$$

If $\vec{\mathbf{F}}$ is conservative then:

$$\vec{\mathbf{F}} = \nabla\phi$$

$$\langle F_1,\ F_2,\ F_3\rangle = \left\langle \frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y},\ \frac{\partial\phi}{\partial z} \right\rangle$$

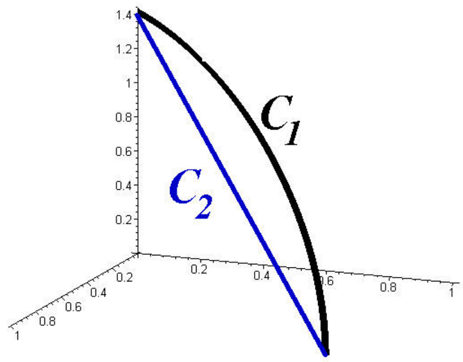
If $\vec{\mathbf{F}}$ is conservative then:

$$\vec{\mathbf{F}} = \nabla\phi$$

$$\langle F_1,\ F_2,\ F_3\rangle = \left\langle \frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y},\ \frac{\partial\phi}{\partial z} \right\rangle$$

$$\frac{\partial\phi}{\partial x} = F_1 \qquad \frac{\partial\phi}{\partial y} = F_2 \qquad \frac{\partial\phi}{\partial z} = F_3$$

Is $\vec{\mathbf{F}} = \langle 0,\ 0,\ -z \rangle = -z\vec{\mathbf{k}}$ conservative?



Is $\vec{\mathbf{F}} = \langle 0,\ 0,\ -z \rangle = -z\vec{\mathbf{k}}$ conservative?

In general:

$$\frac{\partial \phi}{\partial x} = F_1 \qquad \frac{\partial \phi}{\partial y} = F_2 \qquad \frac{\partial \phi}{\partial z} = F_3$$

For this vector field:

$$\frac{\partial \phi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial z} = -z$$

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If $\frac{\partial \phi}{\partial x} = 0$ then ϕ doesn't depend on x
If $\frac{\partial \phi}{\partial y} = 0$ then ϕ doesn't depend on y

$$\phi = \phi(z)$$

$$\frac{\partial \phi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial z} = -z$$

If $\phi = \phi(z)$ then:

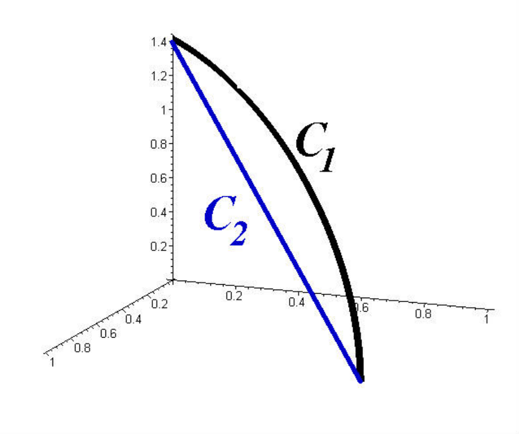
$$\frac{\partial \phi}{\partial z} = -z$$

$$\phi'(z) = -z$$

$$\phi(z) = \int -z \, dz = -\frac{1}{2} z^2 + C$$

$$\vec{\mathbf{F}} = -z\vec{\mathbf{k}} \qquad\qquad \phi = -\frac{1}{2}z^2 + C$$

Integrate $\vec{\mathbf{F}} = -z\vec{\mathbf{k}}$ from $(0,\ 0,\ 1)$ to $(1,\ 1,\ 0)$



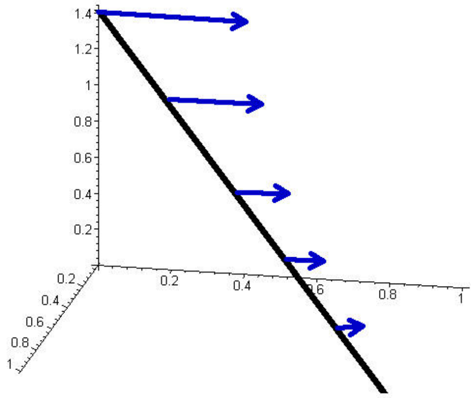
$$\vec{\mathbf{F}} = -z\vec{\mathbf{k}} \qquad\qquad \phi = -\frac{1}{2}z^2 + C$$

Integrate $\vec{\mathbf{F}} = -z\vec{\mathbf{k}}$ from $(0,\ 0,\ 1)$ to $(1,\ 1,\ 0)$

$$\begin{aligned}
\int_{(0,0,1)}^{(1,1,0)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \phi(1,1,0) - \phi(0,0,1) \\
&= \left(-\frac{1}{2}0^2 + C\right) - \left(-\frac{1}{2}1^2 + C\right) \\
&= \frac{1}{2}
\end{aligned}$$

$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = z\vec{\mathbf{j}}$$

Is there a function ϕ such that $\vec{\mathbf{F}} = \nabla\phi$?



$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = z\vec{\mathbf{j}}$$

Is there a function ϕ such that $\vec{\mathbf{F}} = \nabla\phi$?

$$\frac{\partial \phi}{\partial x} = F_1 \qquad \frac{\partial \phi}{\partial y} = F_2 \qquad \frac{\partial \phi}{\partial z} = F_3$$

For this vector field:

$$\frac{\partial \phi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial y} = z \qquad \frac{\partial \phi}{\partial z} = 0$$

$$\frac{\partial \phi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial y} = z \qquad \frac{\partial \phi}{\partial z} = 0$$

If $\frac{\partial \phi}{\partial x} = 0$ then ϕ doesn't depend on x
If $\frac{\partial \phi}{\partial z} = 0$ then ϕ doesn't depend on z

$$\phi = \phi(y)$$

$$\frac{\partial \phi}{\partial y} = \phi'(y)$$

$$\frac{\partial \phi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial y} = z \qquad \frac{\partial \phi}{\partial z} = 0$$

$$\phi = \phi(y)$$

$$\frac{\partial \phi}{\partial y} = \phi'(y) = z$$

Contradiction! Therefore there is no function ϕ so that $\vec{\mathbf{F}} = \nabla \phi$

There is a derivative test to check if $\vec{\mathbf{F}}$ is conservative

$$\frac{\partial \phi}{\partial x} = F_1 \qquad \frac{\partial \phi}{\partial y} = F_2 \qquad \frac{\partial \phi}{\partial z} = F_3$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)$$

Therefore,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

If $\vec{\mathbf{F}}$ is conservative then:

$$\frac{\partial \phi}{\partial x} = F_1 \qquad \frac{\partial \phi}{\partial y} = F_2 \qquad \frac{\partial \phi}{\partial z} = F_3$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)$$

$$\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right)$$

Therefore,

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

If $\vec{\mathbf{F}}$ is conservative then:

$$\frac{\partial \phi}{\partial x} = F_1 \qquad \frac{\partial \phi}{\partial y} = F_2 \qquad \frac{\partial \phi}{\partial z} = F_3$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)$$

$$\frac{\partial F_3}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)$$

Therefore,

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

So if $\vec{\mathbf{F}}$ is conservative, the coordinates must satisfy all of the following conditions:

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

So if $\vec{\mathbf{F}}$ is conservative, the coordinates must satisfy all of the following conditions:

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Rewrite as differences:

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0 \qquad \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} = 0 \qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$
\begin{aligned}
\nabla \times \vec{\mathbf{F}} &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \vec{\mathbf{k}} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \vec{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{\mathbf{k}}
\end{aligned}
$$

Partial derivative conditions if $\vec{\mathbf{F}}$ is conservative:

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0 \qquad \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} = 0 \qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

Compare to the coordinates of $\nabla \times \vec{\mathbf{F}}$:

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{\mathbf{k}}$$

Conclusion: If $\vec{\mathbf{F}}$ is conservative then $\nabla \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$

Test if $\vec{\mathbf{F}} = \langle y - x,\ z - y,\ x - z \rangle$ is conservative

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - x & z - y & x - z \end{vmatrix}$$

Test if $\vec{\mathbf{F}} = \langle y - x,\ z - y,\ x - z \rangle$ is conservative

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - x & z - y & x - z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - y & x - z \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y - x & x - z \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y - x & z - y \end{vmatrix} \vec{\mathbf{k}}$$

Test if $\vec{\mathbf{F}} = \langle y-x,\ z-y,\ x-z\rangle$ is conservative

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-x & z-y & x-z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x-z \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y-x & x-z \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y-x & z-y \end{vmatrix} \vec{\mathbf{k}}$$

$$= (0-1)\,\vec{\mathbf{i}} - (1-0)\,\vec{\mathbf{j}} + (0-1)\,\vec{\mathbf{k}}$$

Conclusion: $\vec{\mathbf{F}}$ is not a conservative vector field.

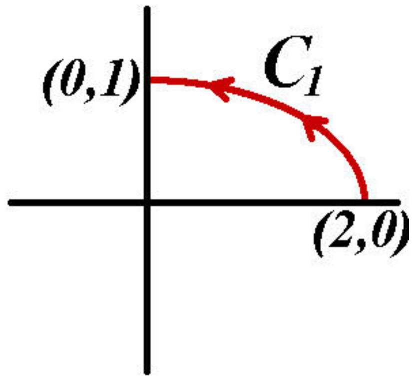
Let $\vec{\mathbf{F}} = (-x + y)\vec{\mathbf{i}} + (x - 2y)\vec{\mathbf{j}}$
Test if $\vec{\mathbf{F}}$ is conservative

Let $\vec{\mathbf{F}} = (-x+y)\vec{\mathbf{i}} + (x-2y)\vec{\mathbf{j}}$
Test if $\vec{\mathbf{F}}$ is conservative

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x+y & x-2y & 0 \end{vmatrix} = \vec{\mathbf{0}}$$

Let $\vec{\mathbf{F}} = (-x + y)\vec{\mathbf{i}} + (x - 2y)\vec{\mathbf{j}}$

Let C_1 be the quarter of the ellipse $\frac{x^2}{4} + y^2 = 1$ connecting $(2,\ 0)$ to $(0,\ 1)$. Calculate $\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$



$\vec{\mathbf{F}} = (-x + y)\vec{\mathbf{i}} + (x - 2y)\vec{\mathbf{j}}$ is conservative so:

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{(2,0)}^{(0,1)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(0,1) - \phi(2,0)$$

$\vec{\mathbf{F}} = (-x + y)\vec{\mathbf{i}} + (x - 2y)\vec{\mathbf{j}}$

$$\frac{\partial \phi}{\partial x} = F_1 \qquad \frac{\partial \phi}{\partial y} = F_2 \qquad \frac{\partial \phi}{\partial z} = F_3$$

For this vector field:

$$\frac{\partial \phi}{\partial x} = -x + y \qquad \frac{\partial \phi}{\partial y} = x - 2y \qquad \frac{\partial \phi}{\partial z} = 0$$

If $\frac{\partial \phi}{\partial z} = 0$ then ϕ doesn't depend of z

$$\phi = \phi(x, y)$$

$$\frac{\partial \phi}{\partial x} = -x + y \qquad \frac{\partial \phi}{\partial y} = x - 2y$$

Let's start with the condition:

$$\frac{\partial \phi}{\partial y} = x - 2y$$

$$(\mathbf{1}) \qquad \frac{\partial \phi}{\partial y} = x - 2y$$

$$(\mathbf{2}) \qquad \phi = \mathbf{???}$$

(1)

$$\frac{\partial \phi}{\partial y} = x - 2y$$

(2)

$$\phi = \textbf{???}$$

$$\phi = \int (x - 2y)\, dy \qquad \text{where } x \text{ is held constant}$$

(1)

$$\frac{\partial \phi}{\partial y} = x - 2y$$

(2)

$$\phi = \textbf{???}$$

$$\begin{aligned} \phi &= \int (x - 2y)\, dy \qquad \text{where } x \text{ is held constant} \\ &= xy - y^2 + \textit{(constant)} \end{aligned}$$

(1) $$\frac{\partial \phi}{\partial y} = x - 2y$$

(2) $$\phi = \textbf{???}$$

$$\begin{aligned} \phi &= \int (x - 2y)\, dy \qquad \text{where } x \text{ is held constant} \\ &= xy - y^2 + \textit{(constant)} \end{aligned}$$

Could be $xy - y^2 + \sin x$

(**1**) $$\frac{\partial \phi}{\partial y} = x - 2y$$

(**2**) $$\phi = \mathbf{???}$$

$$\phi = \int (x - 2y)\, dy \qquad \text{where } x \text{ is held constant}$$
$$= xy - y^2 + \textit{(constant)}$$

Could be $xy - y^2 + e^x$

(1)

$$\frac{\partial \phi}{\partial y} = x - 2y$$

(2)

$$\phi = \textbf{???}$$

$$\begin{aligned} \phi &= \int (x - 2y)\, dy \qquad \text{where } x \text{ is held constant} \\ &= xy - y^2 + G(x) \end{aligned}$$

$$\phi(x,y) = xy - y^2 + G(x)$$

$$\frac{\partial \phi}{\partial x} = -x + y$$

$$\frac{\partial}{\partial x}\left(xy - y^2 + G(x)\right) = -x + y$$

$$\phi(x, y) = xy - y^2 + G(x)$$

$$\frac{\partial \phi}{\partial x} = -x + y$$

$$\frac{\partial}{\partial x}\left(xy - y^2 + G(x)\right) = -x + y$$

$$y - 0 + G'(x) = -x + y$$

$$G'(x) = -x$$

$$G(x) = -\frac{1}{2}x^2 + C$$

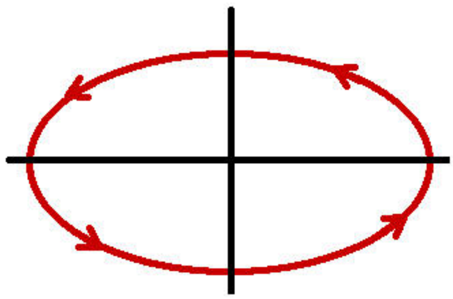
$$\phi(x, y) = xy - y^2 - \frac{1}{2}x^2 + C$$

$$\vec{\mathbf{F}} = (-x+y)\vec{\mathbf{i}} + (x-2y)\vec{\mathbf{j}} \qquad \phi(x,y) = xy - y^2 - \frac{1}{2}x^2 + C$$

$$\begin{aligned}\int_{(2,0)}^{(0,1)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \phi(0,1) - \phi(2,0) \\ &= (0 - 1 - 0 + C) - \left(0 - 0 - \frac{1}{2} \cdot 2^2 + C\right) \\ &= 1\end{aligned}$$

$$\vec{\mathbf{F}} = (-x+y)\vec{\mathbf{i}} + (x-2y)\vec{\mathbf{j}} \qquad \phi(x,y) = xy - y^2 - \frac{1}{2}x^2$$

Let Γ be the entire elliptical path. Calculate $\int_\Gamma \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$



$$\vec{\mathbf{F}} = (-x+y)\vec{\mathbf{i}} + (x-2y)\vec{\mathbf{j}} \qquad \phi(x,y) = xy - y^2 - \frac{1}{2}x^2$$

$$\oint_{\Gamma} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{(2,0)}^{(2,0)} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(2,0) - \phi(2,0) = 0$$