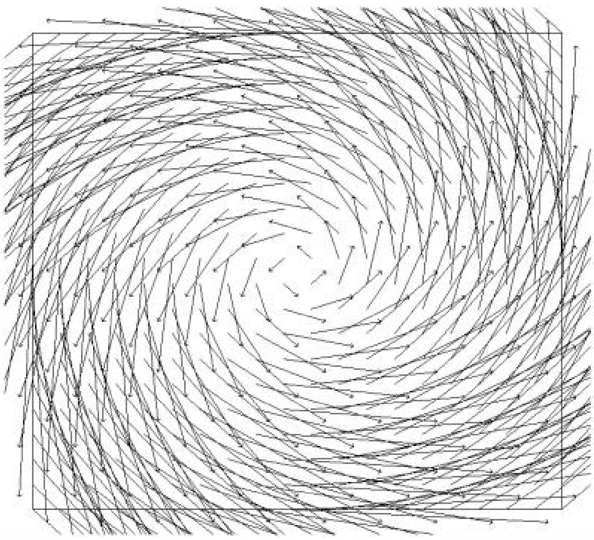
Closed Loop Integrals and $\nabla \times \vec{\mathbf{F}}$

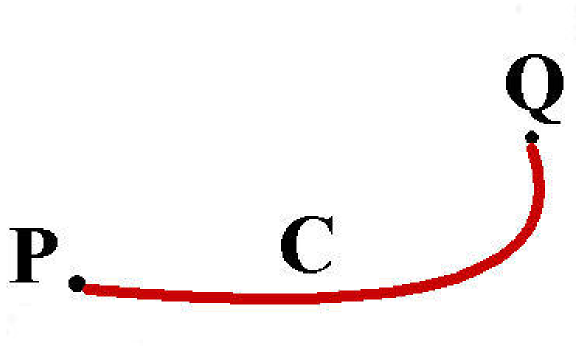
Dr. Elliott Jacobs

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



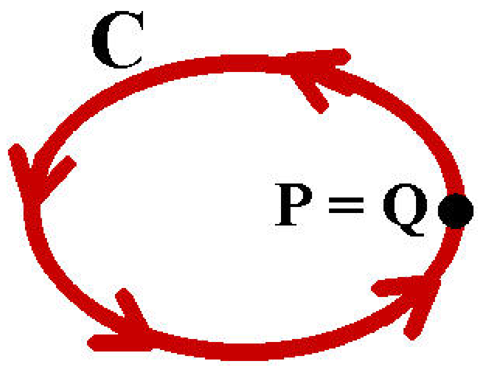
If $\vec{\mathbf{F}}$ is conservative, it's line integral between two points $\mathbf{P}$ and $\mathbf{Q}$ is path independent

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\mathbf{Q}) - \phi(\mathbf{P})$$

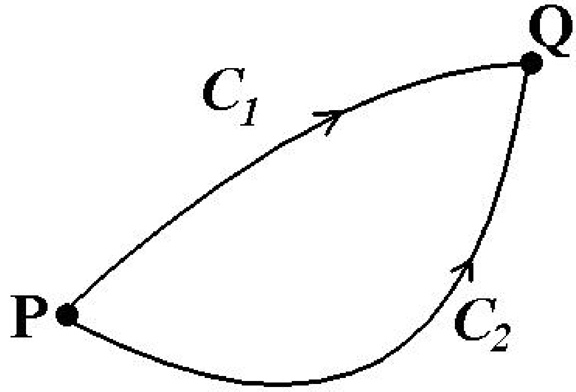


If the path C is a *closed loop* then $\mathbf{P} = \mathbf{Q}$ and the integral is 0.

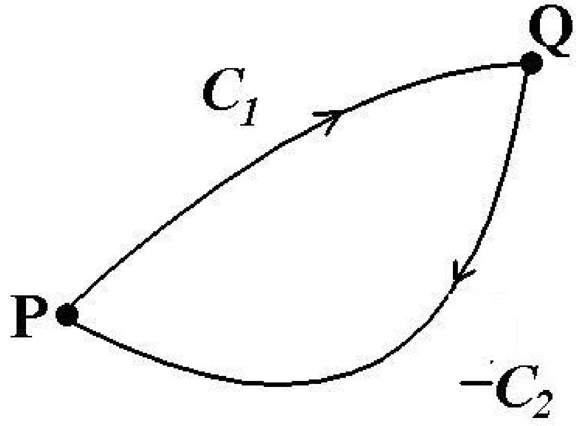
$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\mathbf{P}) - \phi(\mathbf{P}) = 0$$



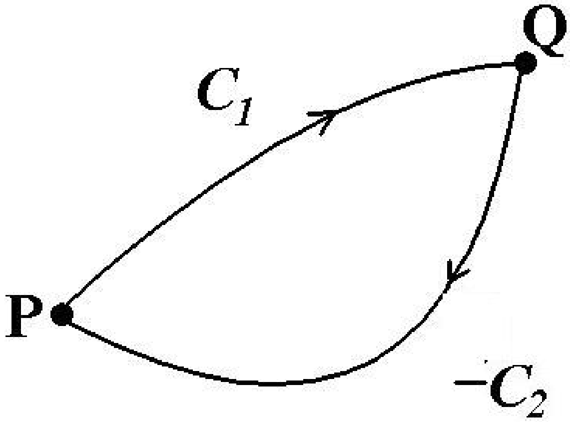
Let C_1 and C_2 be two different paths connecting **P** and **Q**



If we reverse direction along path C_2, we get a closed loop C.



$$
\begin{aligned}
\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{-C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \\
&= \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} - \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}
\end{aligned}
$$

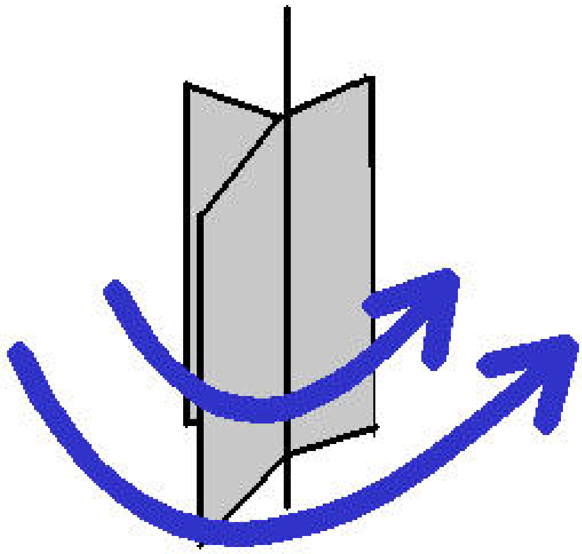


$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} - \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

Thus, line integrals of $\vec{\mathbf{F}}$ are path independent if and only if all closed loop of $\vec{\mathbf{F}}$ are 0.

Application to Fluids

In fluid dynamics, $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ is called the *circulation* of the vector field

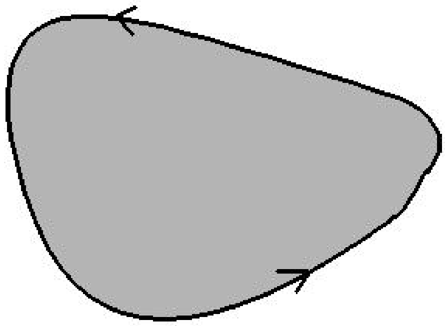


If $\vec{\mathbf{v}}$ is the velocity vector field of an ideal fluid then the circulation is zero.

$$\oint_C \vec{\mathbf{v}} \bullet d\vec{\mathbf{r}} = 0$$

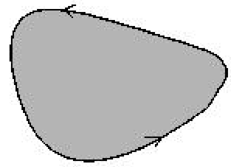
Circulation Per Unit Area

$$\frac{1}{\text{Area}} \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

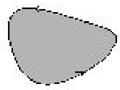


$$\frac{1}{\text{Area}} \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

The circulation per unit area is called the *circulation density.* To find this around one point, take a small region around this point and integrate around the boundary.



Keep making the area smaller. The limit is the circulation per unit area at a point.



$$\lim_{\text{Area}\to 0} \frac{1}{\text{Area}} \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

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There is a formula for this limit in terms of the partial derivatives of the coordinates of $\vec{\mathbf{F}}$

Some preliminaries:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \end{aligned}$$

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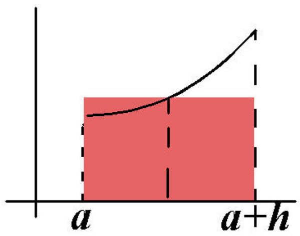
$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} \\
&= \lim_{h \to 0} \frac{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}{h}
\end{aligned}
$$

$$f'(x) \approx \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x}$$

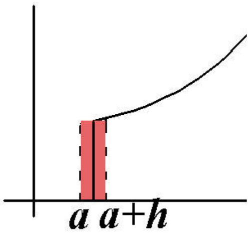
We can approximate the area under a curve by taking the height in the middle and multiplying by the length at the base.

$$\int_a^{a+h} f(x)\,dx \approx f\left(a+\frac{h}{2}\right)\cdot h$$

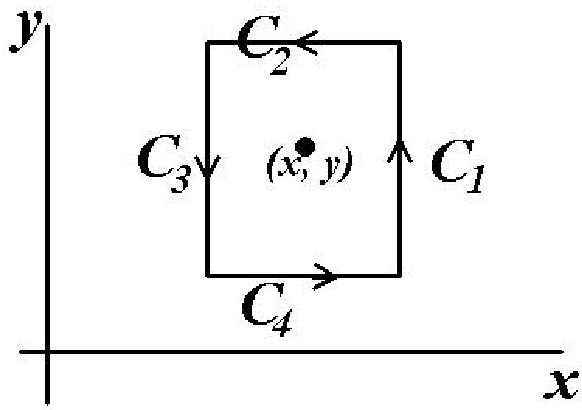


The approximation improves when h is small.

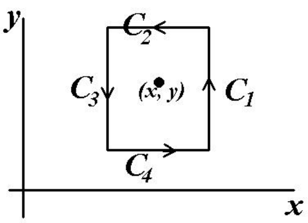
$$\int_a^{a+h} f(x)\,dx \approx f\left(a + \frac{h}{2}\right) \cdot h$$



Let C be a closed rectangular loop in the xy plane. The height of this rectangle is Δy and the base is Δx. There is a point (x, y) in the center of this loop.



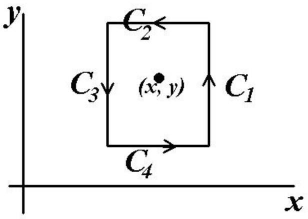
$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_3} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_4} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



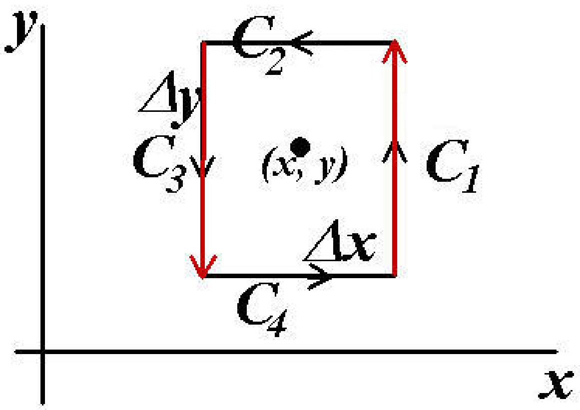
Coordinates of the 4 corners of this rectangle:

$$\left(x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}\right) \qquad \left(x + \frac{\Delta x}{2}, y - \frac{\Delta y}{2}\right)$$

$$\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}\right) \qquad \left(x - \frac{\Delta x}{2}, y + \frac{\Delta y}{2}\right)$$

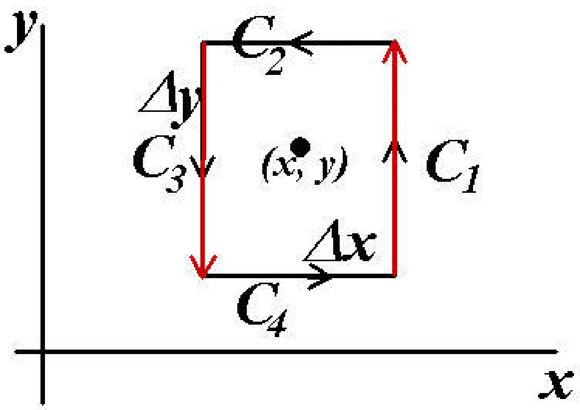


Let's start with path C_1



$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} F_1\,dx + F_2\,dy = \int_{C_1} F_2\,dy$$

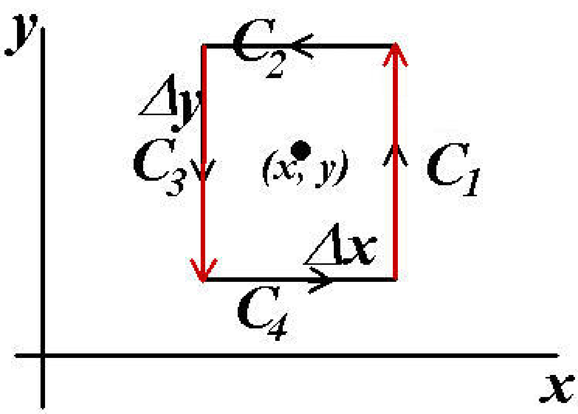
Let's start with path C_1



$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} F_2 \, dy \approx F_2 \left(x + \frac{\Delta x}{2}, \; y \right) \cdot \Delta y$$

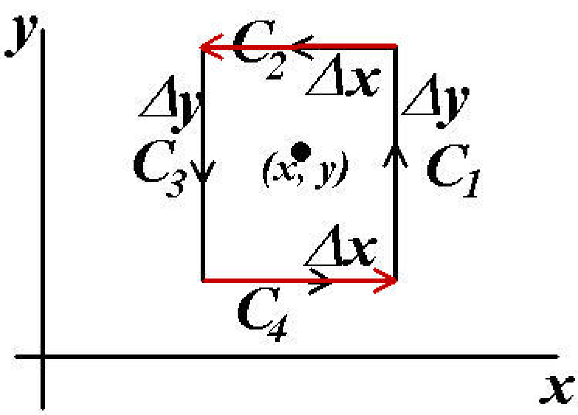
$$\int_{C_3} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_3} F_2 \, dy$$

$$\approx -F_2\left(x - \frac{\Delta x}{2}, y\right) \cdot \Delta y$$



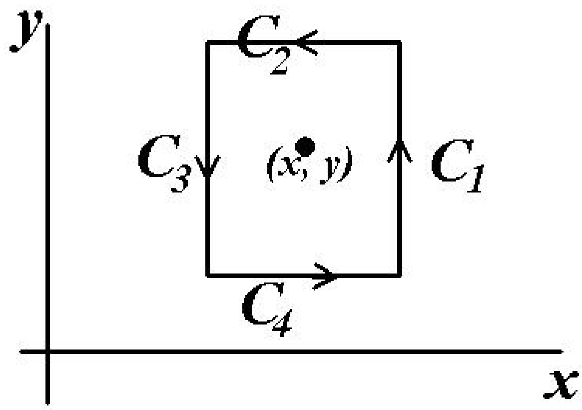
$$
\begin{aligned}
&\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_3} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \\
&\approx F_2\left(x + \frac{\Delta x}{2}, y\right)\Delta y - F_2\left(x - \frac{\Delta x}{2}, y\right)\Delta y \\
&= \left(\frac{F_2\left(x + \frac{\Delta x}{2}, y\right)\Delta y - F_2\left(x - \frac{\Delta x}{2}, y\right)}{\Delta x}\right)\Delta x\,\Delta y \\
&\approx \frac{\partial F_2}{\partial x}\,\Delta x\,\Delta y
\end{aligned}
$$

Next, we repeat this argument for paths C_2 and C_4



$$\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_4} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \approx -\frac{\partial F_1}{\partial y}\, \Delta x\, \Delta y$$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \sum_{i=1}^{4} \int_{C_i} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



$$\oint \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \approx \frac{\partial F_2}{\partial x}\,\Delta x\,\Delta y - \frac{\partial F_1}{\partial y}\,\Delta x\,\Delta y$$

$$\frac{1}{\text{Area}} \oint \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \approx \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

The error in this approximation goes to 0 as Δx and Δy go to 0.

$$\lim_{\text{Area}\to 0} \frac{1}{\text{Area}} \oint \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

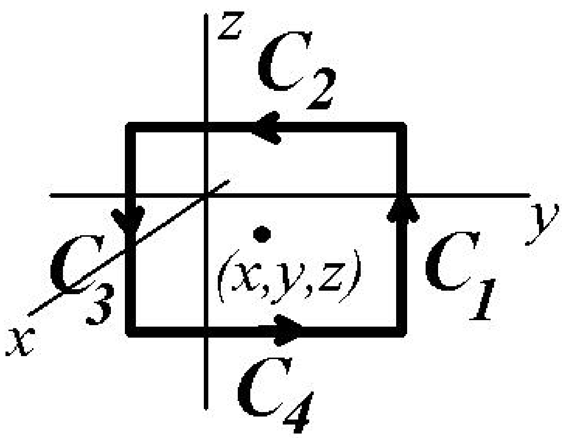
So, the significance of the expression:

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

is that it represents the circulation density at (x, y) in a plane parallel to the xy plane.

We could have taken our region to be a rectangle parallel to the yz plane

$$\lim_{\text{Area}\to 0} \frac{1}{\text{Area}} \oint \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$$



$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \vec{\mathbf{k}}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \vec{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{\mathbf{k}}$$

Circulation density at a point in the yz plane:

$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{i}} = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$$

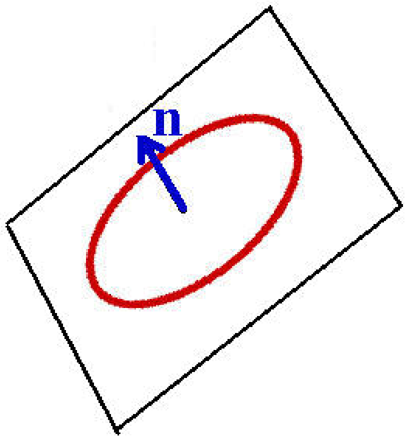
Circulation density at a point in the xz plane:

$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{j}} = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}$$

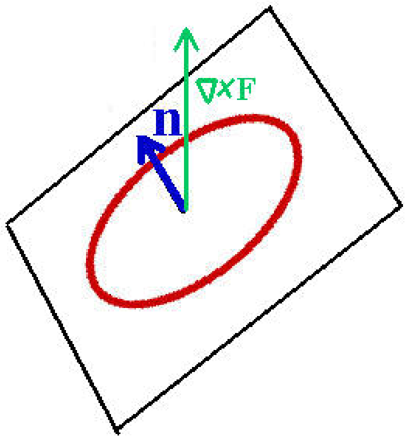
Circulation density at a point in the xy plane:

$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{k}} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Let $\vec{\mathbf{n}}$ be the unit normal vector to a closed loop in any plane.



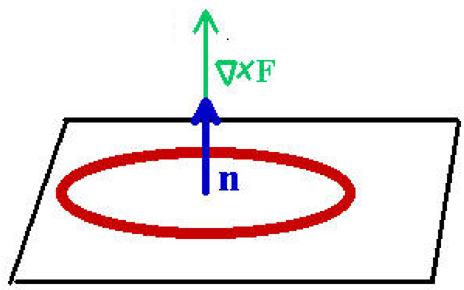
$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}$ is the circulation density in the plane perpendicular to $\vec{\mathbf{n}}$



$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}} = |\nabla \times \vec{\mathbf{F}}| \cos \theta$$

This is maximized with $\theta = 0$ in which case:

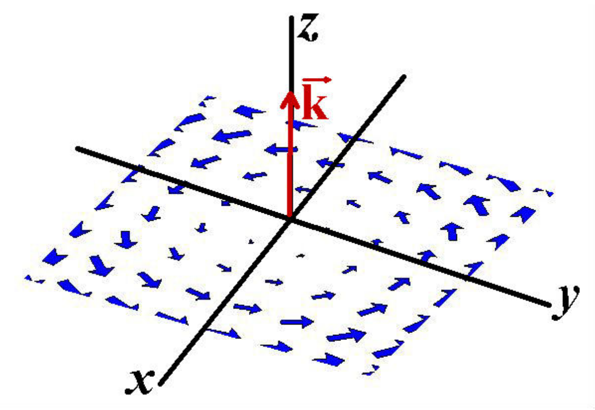
$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}} = |\nabla \times \vec{\mathbf{F}}|$$



Particle orbiting in a circle of radius a with an angular velocity of ω

$$\vec{\mathbf{r}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} = (a\cos\omega t)\vec{\mathbf{i}} + (a\sin\omega t)\vec{\mathbf{j}}$$

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = (-\omega a\sin\omega t)\vec{\mathbf{i}} + (\omega a\cos\omega t)\vec{\mathbf{j}} = -\omega y\vec{\mathbf{i}} + \omega x\vec{\mathbf{j}}$$



$$\vec{\mathbf{v}} = -\omega y\vec{\mathbf{i}} + \omega x\vec{\mathbf{j}}$$

$$\nabla \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\vec{\mathbf{k}}$$