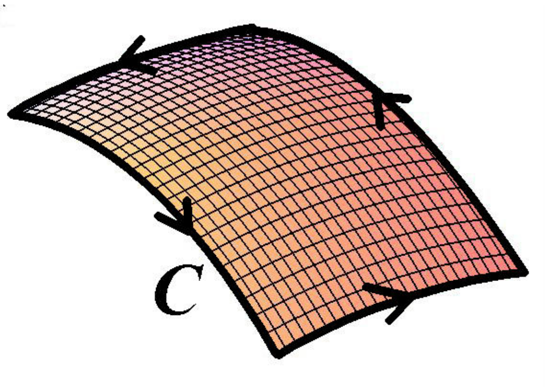
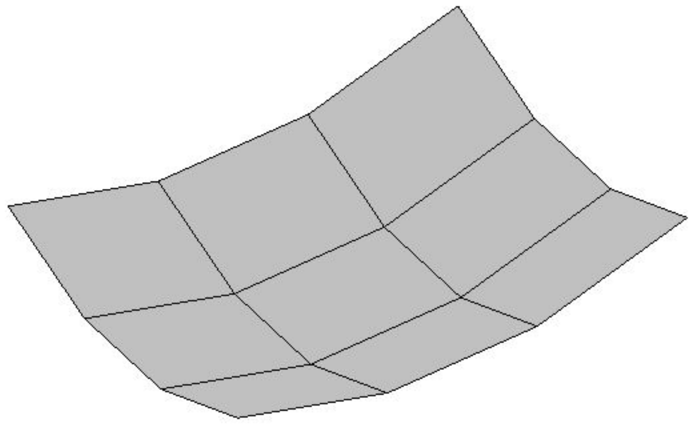
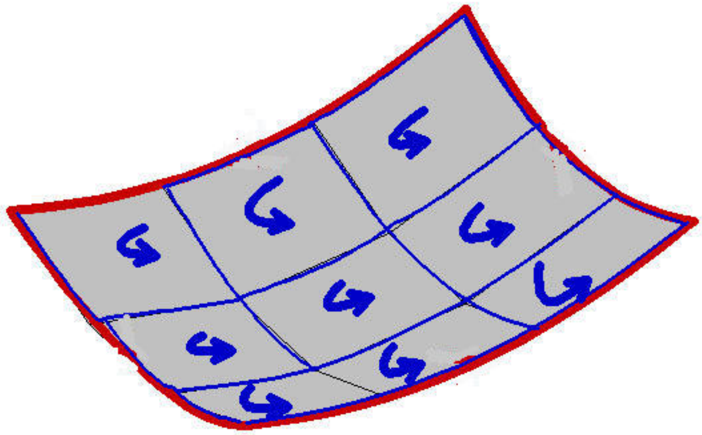
Stokes' Theorem

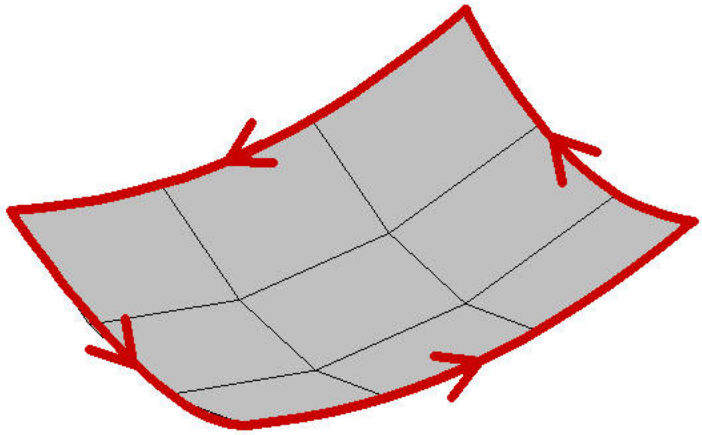
Dr. Elliott Jacobs



$$\iint_S (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\, dS$$



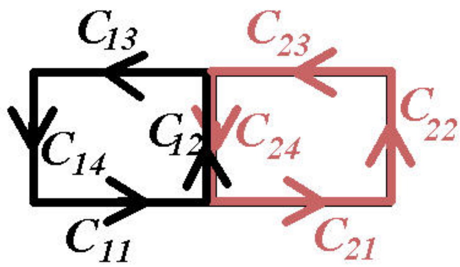




Calculate the circulation around both sections:

$$\int_{C_{11}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{12}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{13}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{14}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

$$\int_{C_{21}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{22}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{23}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{24}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



Calculate the circulation around both sections:

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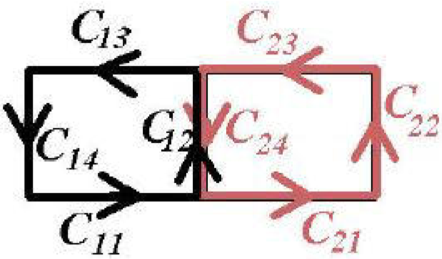
$$\int_{C_{21}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{22}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{23}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{C_{24}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



Calculate the circulation around both sections:

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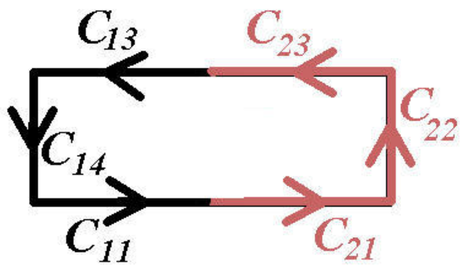
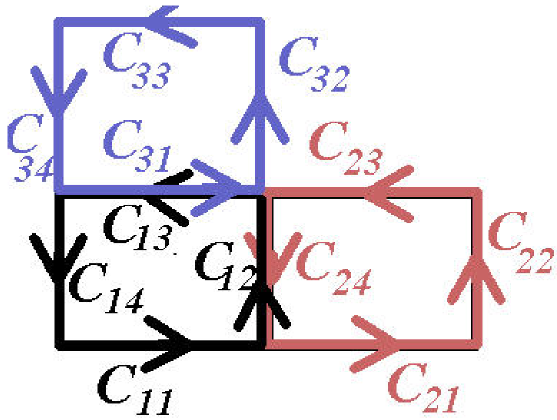
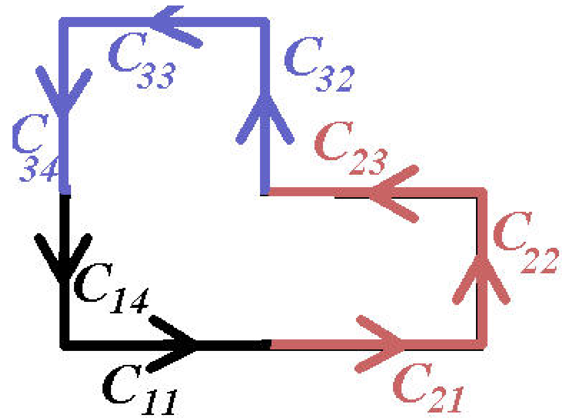
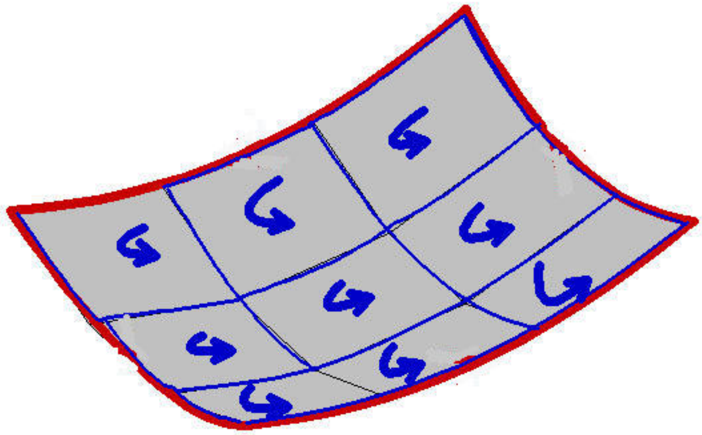
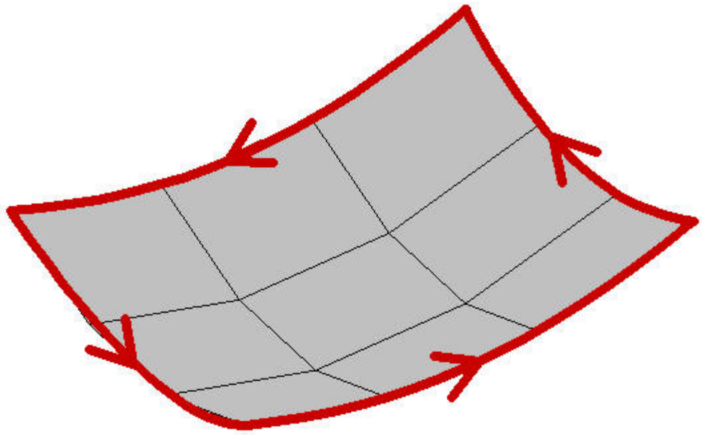




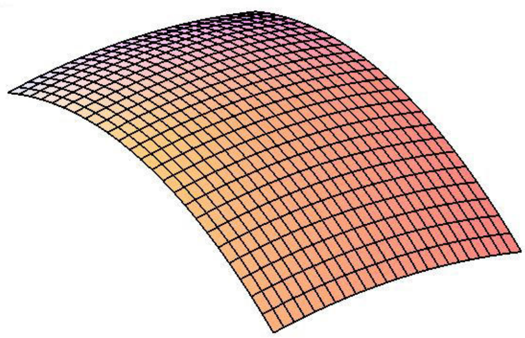
C33
C32
C34
C31
C23
C13
C12
C24
C22
C14
C11
C21


C_{33}
C_{32}
C_{34}
C_{23}
C_{22}
C_{14}
C_{11}
C_{21}

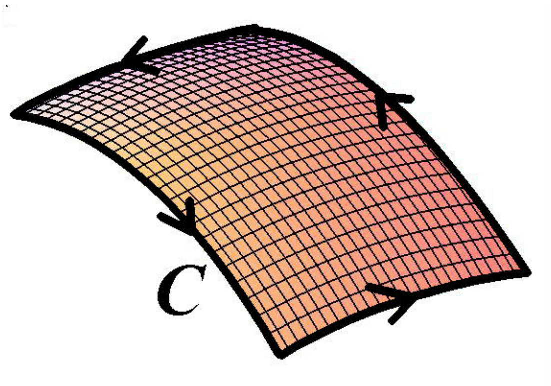




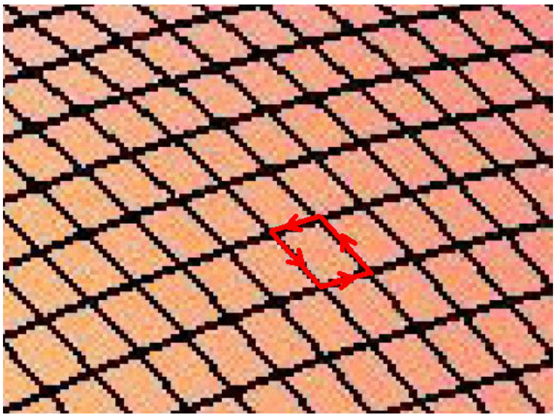
$$\sum_{i=1}^{n} \oint_{C_i} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



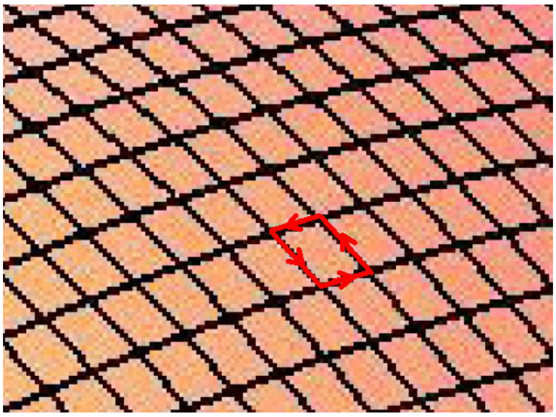
$$\sum_{i=1}^{n} \oint_{C_i} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \oint_{C} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



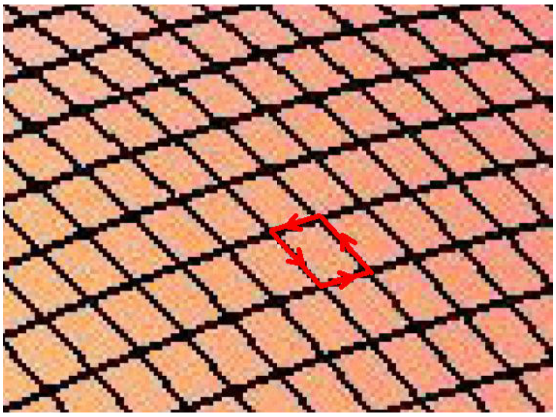
$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}_i \text{ is circulation per unit area at a point}$$



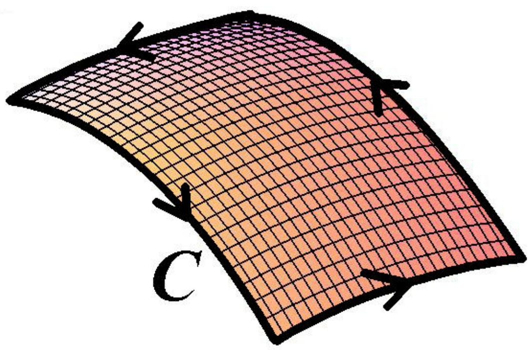
$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}_i \approx \frac{1}{\Delta S_i} \oint_{C_i} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



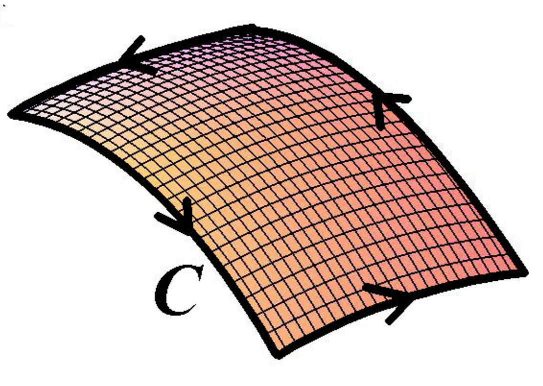
$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}_i \, \Delta S_i \approx \oint_{C_i} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



$$\sum_{i=1}^{n}(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}_i \, \Delta S_i \approx \sum_{i=1}^{n} \oint_{C_i} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \oint_{C} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

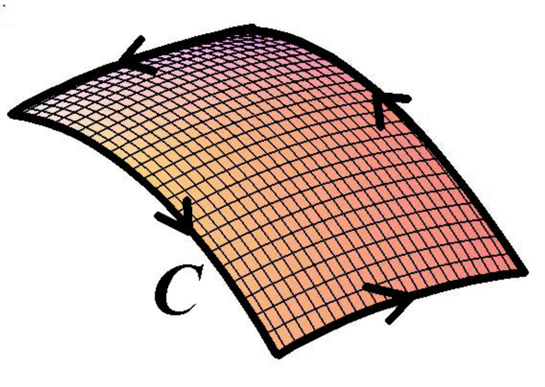


$$\lim_{n\to\infty}\sum_{i=1}^{n}(\nabla\times\vec{\mathbf{F}})\bullet\vec{\mathbf{n}}_i\,\Delta S_i=\oint_C\vec{\mathbf{F}}\bullet d\vec{\mathbf{r}}$$



Stokes' Theorem

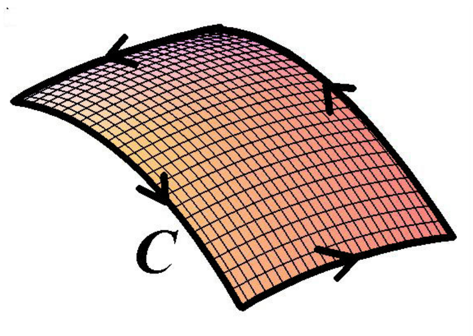
$$\iint_S (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\, dS = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



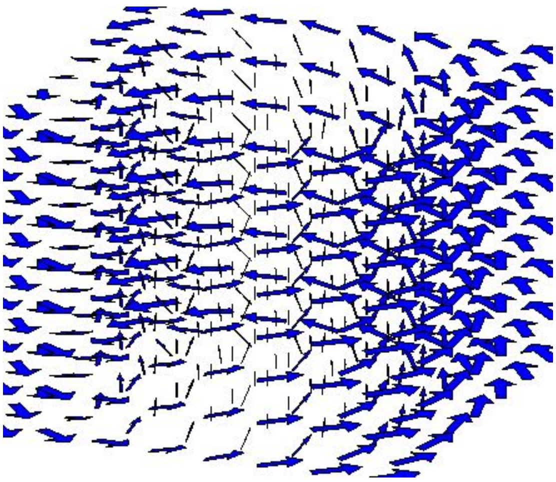
Conditions for Stokes' Theorem

$$\iint_S (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\, dS = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

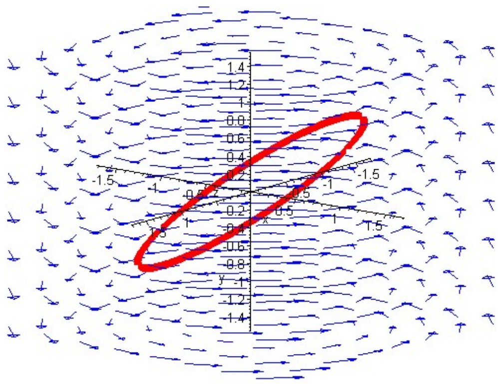
1. C is a closed loop forming the boundary of S
2. $\nabla \times \vec{\mathbf{F}}$ must exist at all points in S



$$\vec{\mathbf{F}} = -y\vec{\mathbf{i}} + x\vec{\mathbf{j}}$$

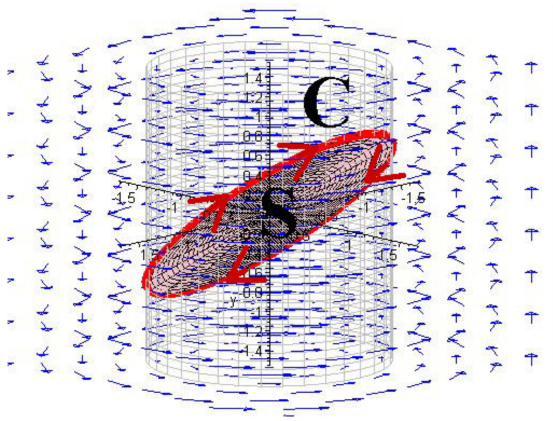


$$\vec{\mathbf{F}} = \langle -y,\ x,\ 0 \rangle$$



$$\vec{\mathbf{F}} = \langle -y,\ x,\ 0\rangle$$

Intersection of plane $z = y$ with cylinder $x^2 + y^2 = 1$



$$\vec{\mathbf{F}} = \langle -y,\ x,\ 0 \rangle$$

Intersection of plane $z = y$ with cylinder $x^2 + y^2 = 1$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\vec{\mathbf{k}}$$

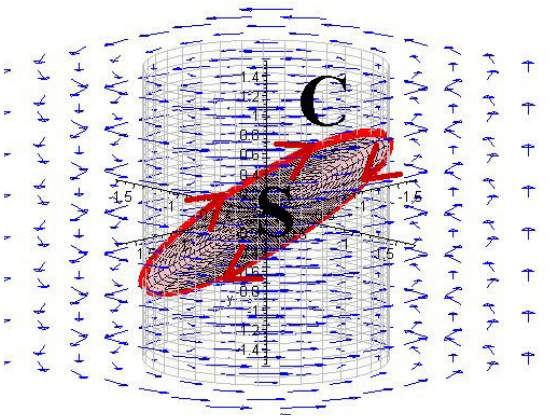
If $\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle x,\ y,\ y\rangle$ then:

$$\vec{\mathbf{n}}\, dS = \left(\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y}\right) dx\, dy = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -\vec{\mathbf{j}} + \vec{\mathbf{k}}$$

$$(\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\, dS = \langle 0,\ 0,\ 2\rangle \bullet \langle 0,\ -1,\ 1\rangle\, dx\, dy = 2\, dx\, dy$$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_S (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} 2\, dx\, dy$$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_{\mathcal{D}} 2\,dx\,dy = 2 \cdot \text{Area}(\mathcal{D}) = 2\pi$$



Comparison: direct calculation of the line integral:

$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle x,\ y,\ y \rangle = \langle \cos\theta,\ \sin\theta,\ \sin\theta \rangle$$

$$d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{d\theta}\, d\theta = \langle -\sin\theta,\ \cos\theta,\ \cos\theta \rangle\, d\theta$$

$$\vec{\mathbf{F}} = \langle -y,\ x,\ 0 \rangle = \langle -\sin\theta,\ \cos\theta,\ 0 \rangle$$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^{2\pi} \left(\sin^2\theta + \cos^2\theta\right)\, d\theta = 2\pi$$