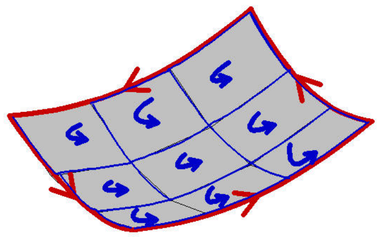
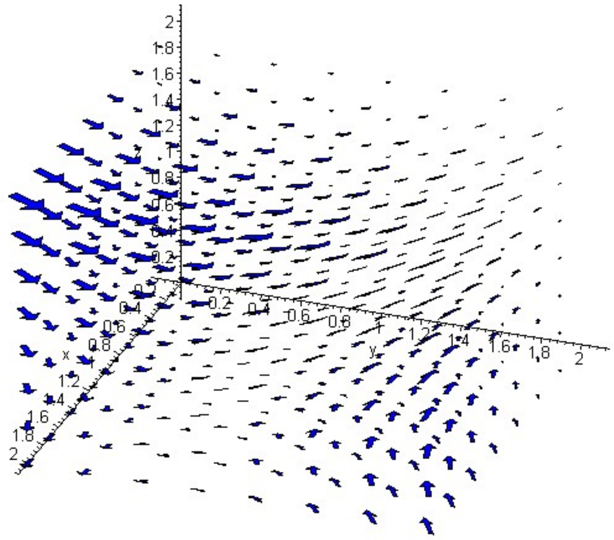
Stokes' Theorem Examples. Green's Theorem

Dr. Elliott Jacobs

$$\iint_S (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\, dS = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

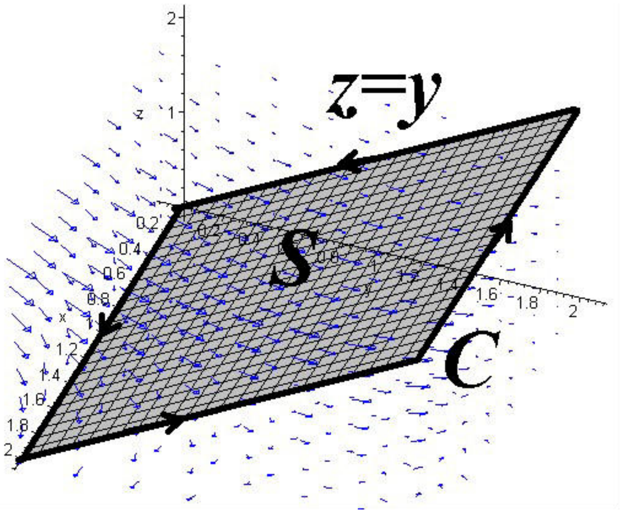


$$\vec{\mathbf{F}} = \langle x,\ xz,\ xy \rangle$$



$$\vec{\mathbf{F}} = \langle x,\ xz,\ xy \rangle$$

$$\text{Calculate } \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



$$\vec{\mathbf{F}} = \langle x,\ xz,\ xy \rangle$$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xz & xy \end{vmatrix} = \langle 0,\ -y,\ z \rangle$$

$$\vec{\mathbf{F}} = \langle x,\ xz,\ xy\rangle$$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xz & xy \end{vmatrix} = \langle 0,\ -y,\ z\rangle$$

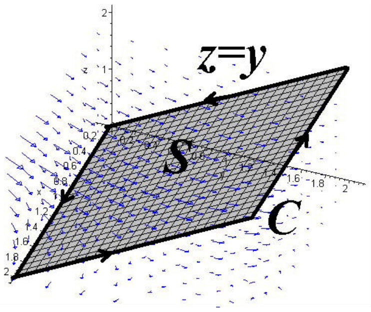
On surface S, $z = y$ so:

$$\nabla \times \vec{\mathbf{F}} = \langle 0,\ -y,\ y\rangle \text{ and } \vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle x,\ y,\ y\rangle$$

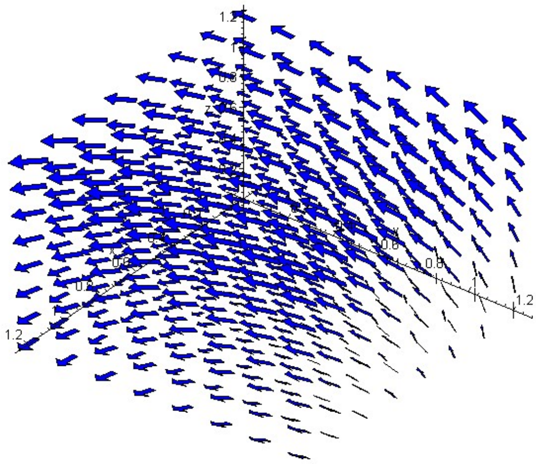
Next, calculate $\vec{\mathbf{n}}\, dS = \left(\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y}\right)\, dx\, dy$

$$\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 0,\ -1,\ 1 \rangle$$

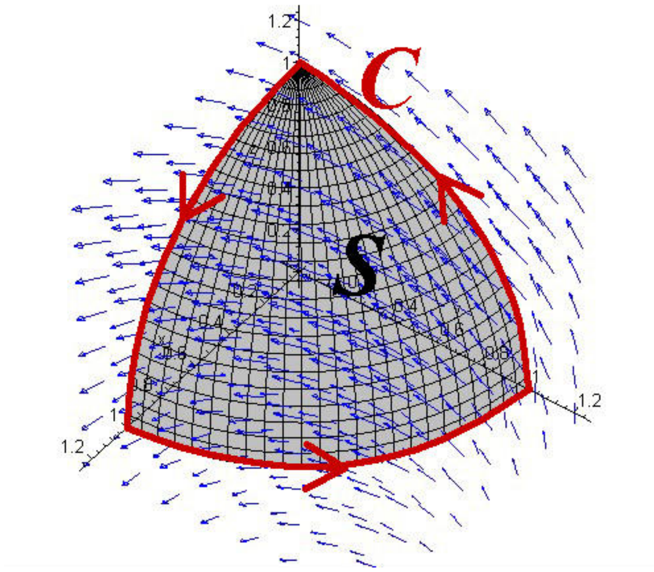
$$
\begin{aligned}
\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \iint_S (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\, dS \\
&= \int_0^2 \int_0^2 \langle 0,\ -y,\ y \rangle \bullet \langle 0,\ -1,\ 1 \rangle \, dx\, dy \\
&= \int_0^2 \int_0^2 2y \, dx\, dy = 8
\end{aligned}
$$



$$\vec{\mathbf{F}} = \langle x,\ -z,\ y\rangle$$



$$\vec{\mathbf{F}} = \langle x,\ -z,\ y \rangle$$



$$\vec{\mathbf{F}} = \langle x,\ -z,\ y \rangle$$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -z & y \end{vmatrix} = \langle 2,\ 0,\ 0 \rangle = 2\vec{\mathbf{i}}$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$$

$$\vec{\mathbf{n}}\,dS = \left(\frac{\partial\vec{\mathbf{r}}}{\partial\phi}\times\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right)\,d\phi\,d\theta$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$$

$$\begin{aligned}\vec{\mathbf{n}}\,dS &= \left(\frac{\partial\vec{\mathbf{r}}}{\partial\phi}\times\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right)d\phi\,d\theta\\ &= \left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi\right\rangle d\phi\,d\theta\end{aligned}$$

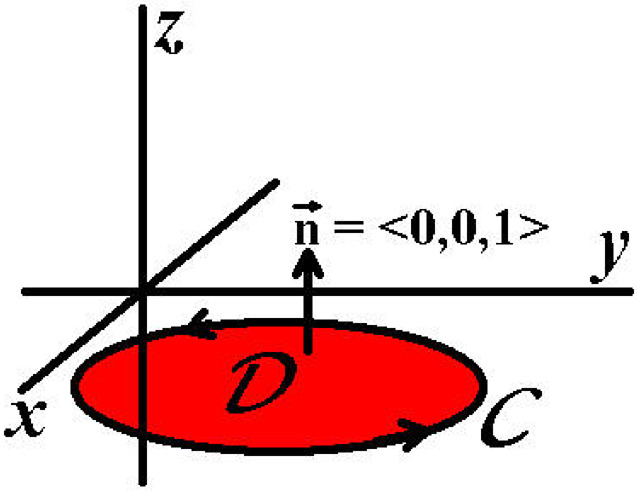
$$\nabla \times \vec{\mathbf{F}} = \langle 2,\ 0,\ 0 \rangle$$

$$\vec{\mathbf{n}}\,dS = \left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi \right\rangle\,d\phi\,d\theta$$

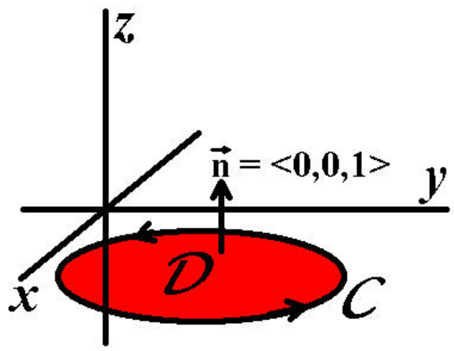
$$\begin{aligned}
\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \iint_S (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{n}}\,dS \\
&= \int_0^{\pi/2}\int_0^{\pi/2} 2\sin^2\phi\cos\theta\,d\phi\,d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

Green's Theorem

Suppose S and it's boundary C are entirely in the xy plane



$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_{\mathcal{D}} (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{k}}\, dS = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$



$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy$$

$$\oint_C F_1 \, dx + F_2 \, dy = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy$$

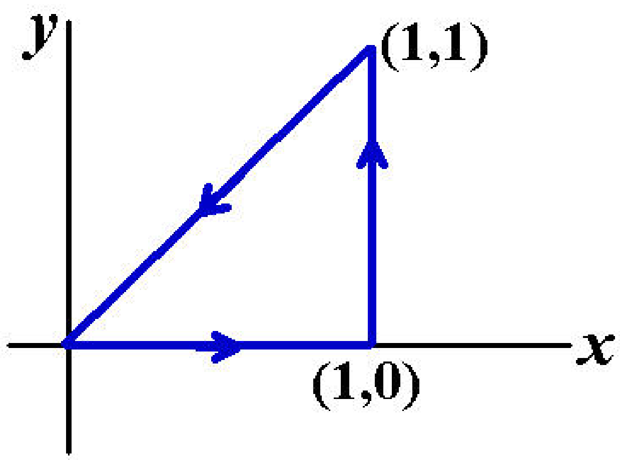
$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy$$

$$\oint_C F_1 \, dx + F_2 \, dy = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy$$

$$\oint_C P \, dx + Q \, dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy$$

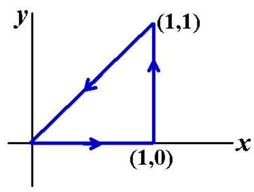
Green's Theorem

$$\vec{\mathbf{F}} = \left\langle x - ye^{x^2},\ e^y,\ 0 \right\rangle \qquad \text{Calculate } \oint_{\mathcal{T}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

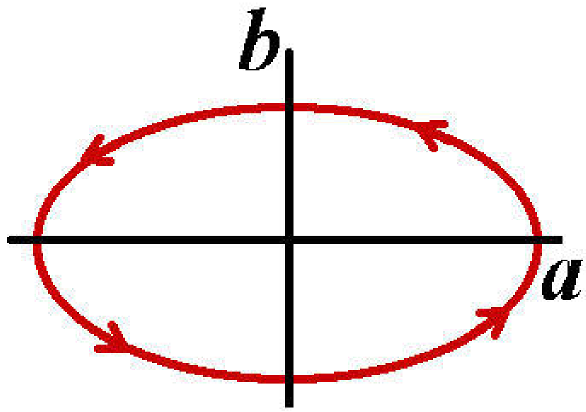


$$P = x - ye^{x^2} \qquad Q = e^y$$

$$\begin{aligned}\oint_C P\,dx + Q\,dy &= \iint_{\mathcal{T}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy \\ &= \int_0^1\int_0^x e^{x^2}\,dy\,dx \\ &= \int_0^1 xe^{x^2}\,dx = \frac{1}{2}(e-1)\end{aligned}$$



With Green's Theorem, it's possible to use a line integral to find the area of a region in the xy plane.



$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

If $P = -y$ and $Q = 0$ then:

$$\oint_C -y\,dx = \iint_{\mathcal{D}} (0 - (-1))\,dx\,dy == \iint_{\mathcal{D}} 1\,dx\,dy = \text{Area}(\mathcal{D})$$

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

If $P = 0$ and $Q = x$ then:

$$\oint_C x\,dy = \iint_{\mathcal{D}} 1\,dx\,dy = \text{Area}(\mathcal{D})$$

$$\oint_C -y\,dx = \text{Area}(\mathcal{D})$$

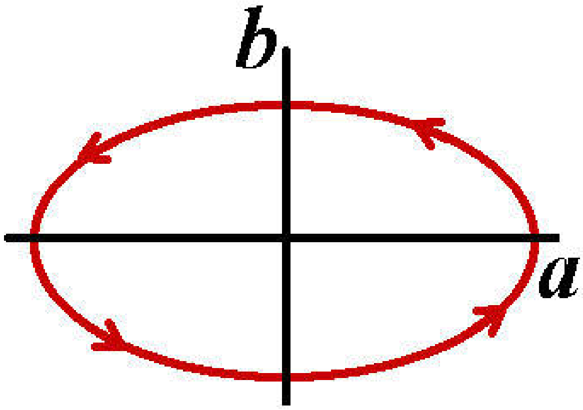
$$\oint_C x\,dy = \text{Area}(\mathcal{D})$$

Add these integrals:

$$\oint_C -y\,dx + x\,dy = 2\text{Area}(\mathcal{D})$$

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(\mathcal{D})$$

$$x = a\cos t \qquad y = b\sin t \qquad \text{where } 0 \le t \le 2\pi$$



Find the area of the ellipse

$$x = a\cos t \qquad y = b\sin t \qquad \text{where } 0 \le t \le 2\pi$$

$$dx = -a\sin t\,dt \qquad dy = b\cos t\,dt$$

$$x = a\cos t \qquad y = b\sin t \qquad \text{where } 0 \le t \le 2\pi$$

$$dx = -a\sin t\,dt \qquad dy = b\cos t\,dt$$

$$\begin{aligned}
\text{Area of Ellipse} &= \frac{1}{2}\oint_C -y\,dx + x\,dy \\
&= \frac{1}{2}\int_0^{2\pi} (-b\sin t)(-a\sin t\,dt) + (a\cos t)(b\cos t\,dt) \\
&= \frac{1}{2}\int_0^{2\pi} \left(ab\sin^2 t + ab\cos^2 t\right)\,dt \\
&= \frac{ab}{2}\int_0^{2\pi} 1\,dt = \pi ab
\end{aligned}$$