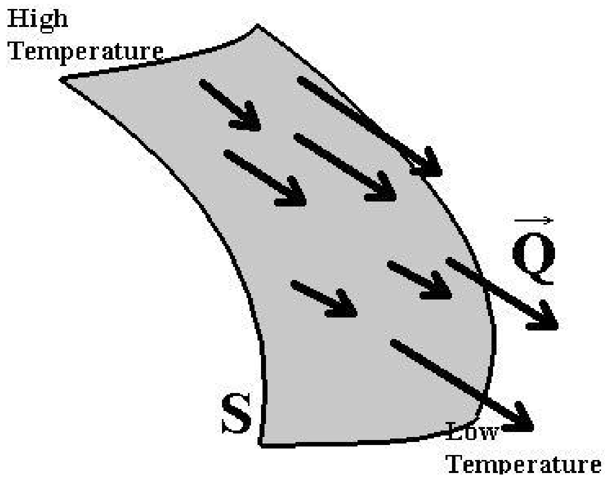
Application of Divergence Theorem
The Heat Equation

Dr. Elliott Jacobs

Let $\vec{\mathbf{Q}}$ denote the heat flow vector



$|\vec{\mathbf{Q}}| = \text{cal/sec/m}^2$

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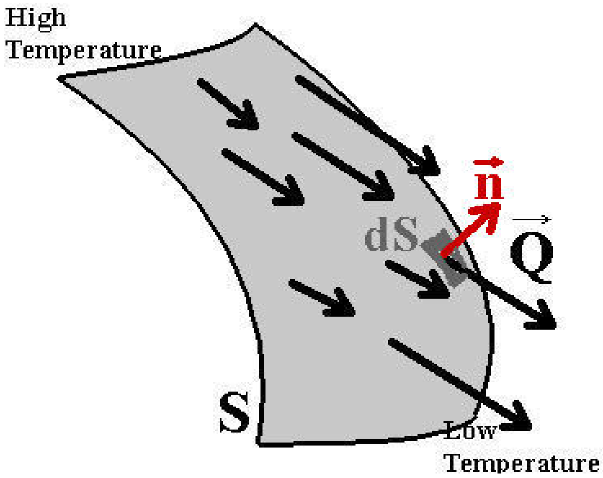

$\vec{\mathbf{Q}} \bullet \vec{\mathbf{n}}\, dS$ is cal/sec of heat flowing through a section of surface of area dS

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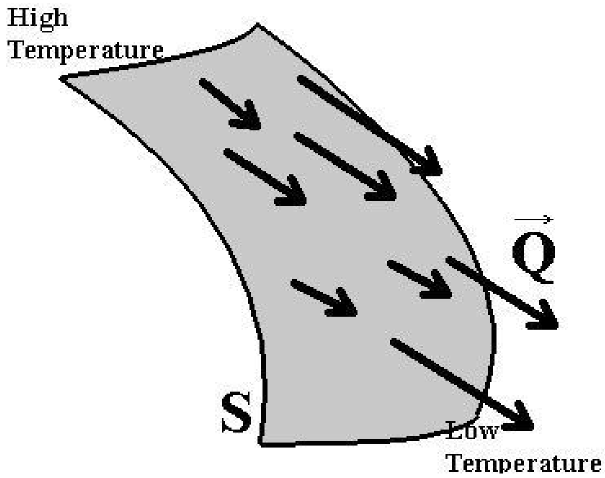
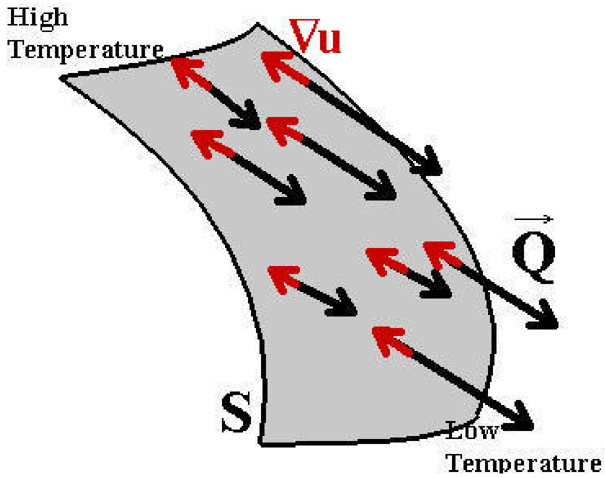


$H = \iint_S \vec{\mathbf{Q}} \bullet \vec{\mathbf{n}}\, dS$ is cal/sec of heat flowing through the entire surface S

Let u be the temperature at point (x, y, z) at time t

$$u = u(x, y, z, t)$$

The temperature gradient points in the direction of increasing temperature



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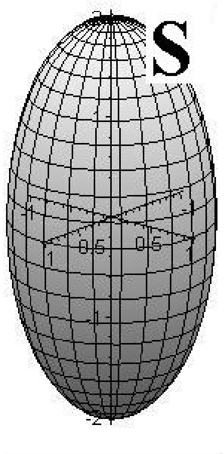
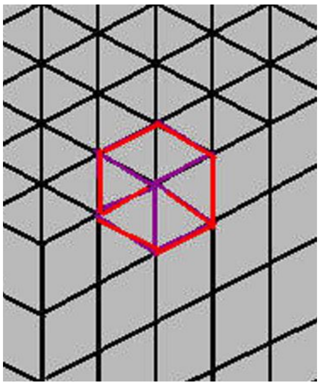


In an interior section of volume ΔV, the heat gain is proportional to the mass and the change in temperature

$$\text{Mass Of Section} = \rho\,\Delta V$$

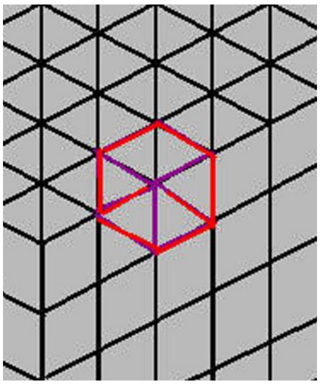
$$\text{Heat Gain} = c \cdot \Delta u \cdot \rho\,\Delta V$$



$$\text{Heat Loss} = -c\rho\Delta u\Delta V$$

$$\text{Rate of Heat Loss} = -c\rho\frac{\Delta u}{\Delta t}\,\Delta V \approx -c\rho\frac{\partial u}{\partial t}\,\Delta V$$

This is the rate that the heat is leaving one small interior section



Total rate that the heat is leaving the solid is:

$$H = \iiint_V -c\rho \frac{\partial u}{\partial t}\, dV$$

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We also know that this heat loss is also given by:

$$H = -k \iint_S \nabla u \bullet \vec{\mathbf{n}} \, dS$$

$$-k \iint_S \nabla u \bullet \vec{\mathbf{n}}\, dS = \iiint_V -c\rho \frac{\partial u}{\partial t}\, dV$$

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$$-k \iiint_V \nabla \bullet (\nabla u) \, dV = \iiint_V -c\rho \frac{\partial u}{\partial t} \, dv$$

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$$0 = \iiint_V \left(k\nabla^2 u - c\rho \frac{\partial u}{\partial t} \right) dV$$

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If this is true for all three dimensional regions V then at all points:

$$0 = k\nabla^2 u - c\rho \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho} \nabla^2 u$$

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Let $\alpha = \sqrt{\frac{k}{c\rho}}$

$$\frac{\partial u}{\partial t} = \alpha^2\nabla^2 u$$

The Heat Equation

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

In the special case where $u = u(x, t)$, the heat equation becomes:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Joseph Fourier