Differential Equations

Differential Equation

A differential equation is an equation that involves the derivative of an unknown function.

$$\frac{dy}{dx} - 2y = 0$$

$$\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} + y = e^x$$

A differential equation could involve higher order derivatives

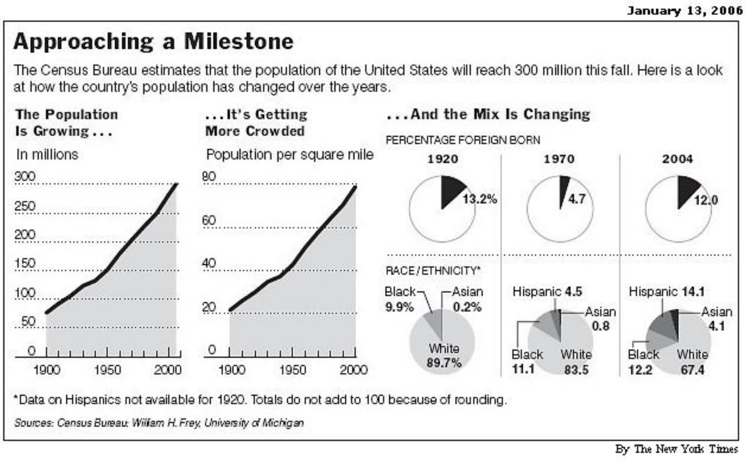
$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

A differential equation could involve partial derivatives

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where } u = u(x,t)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \qquad \text{where } \Psi = \Psi(x,t)$$

Let $P = P(t)$ be the population at time t.



Malthusian Model of Growth

Population grows at a rate proportional to the size of the population at any point in time.

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$$\frac{dP}{dt} = kP$$

Let $A = A(t)$ be the amount of money in a bank account after t years. The bank pays 1 percent interest. Find a formula for $A(t)$.

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$$dA = A \cdot 0.01 \cdot dt$$

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$$\text{Interest} = (\text{Principal})(\text{rate})(\text{time})$$

$$dA = A \cdot (0.01) \cdot dt$$

$$\frac{dA}{dt} = 0.01A$$

Population problem:

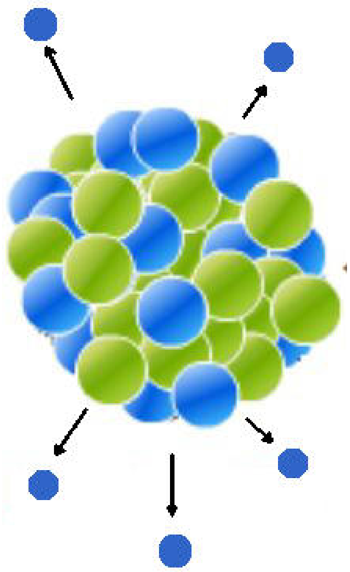
$$\frac{dP}{dt} = kP$$

Bank account problem:

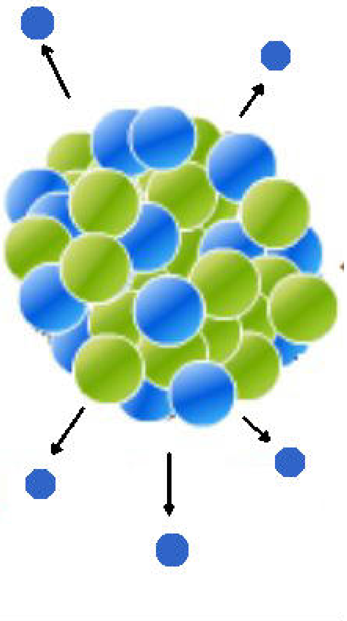
$$\frac{dA}{dt} = 0.01A$$

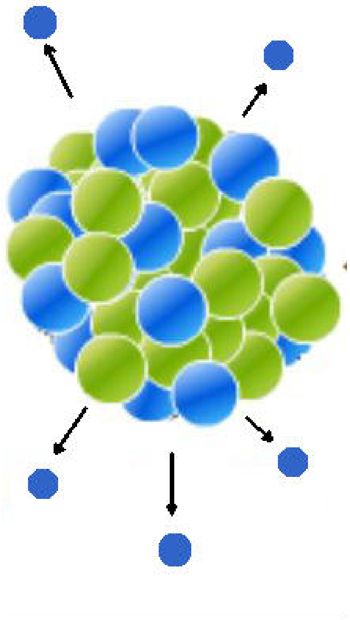
The mass of a radioactive object decreases with time:

$$M = M(t)$$

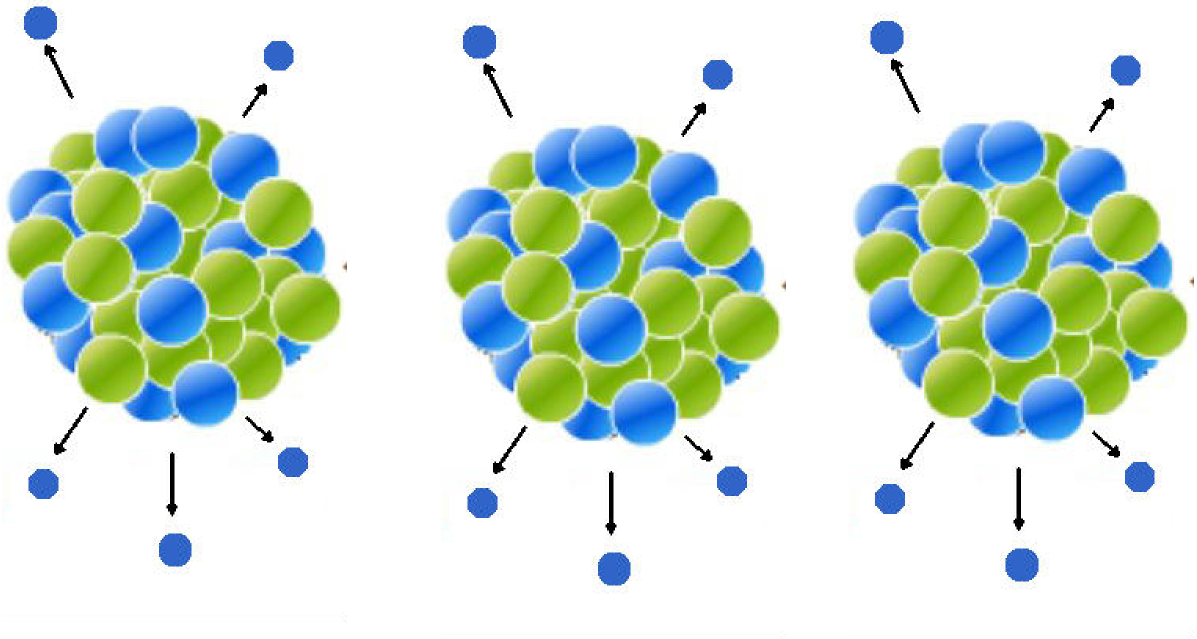


If we double the mass, we double the rate at which the mass decreases.

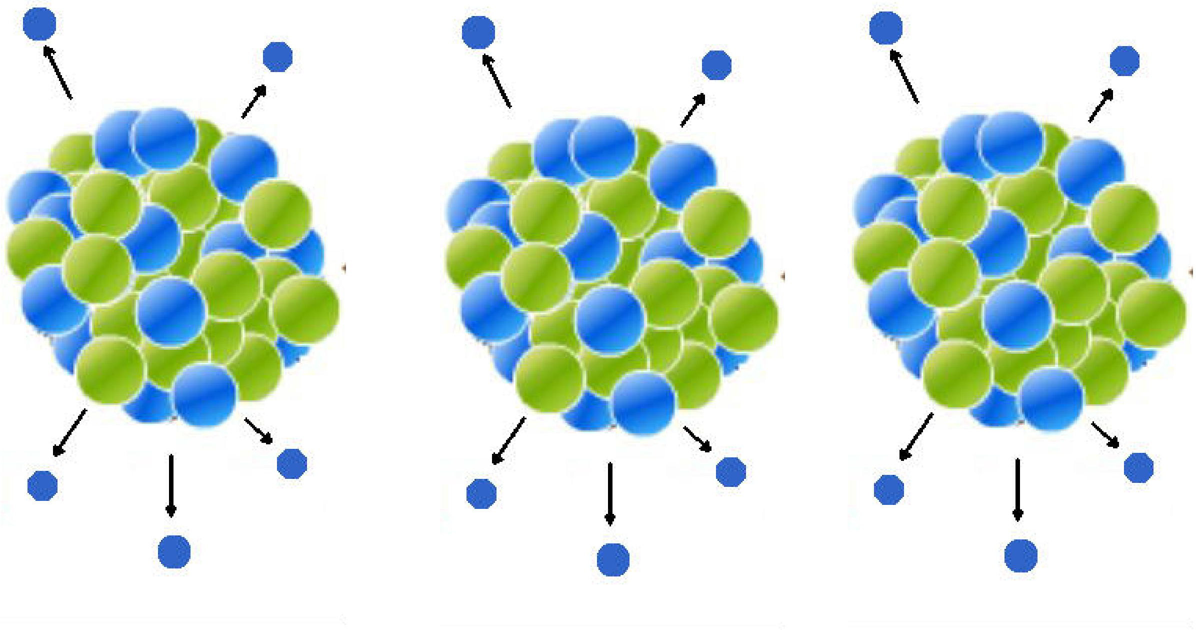




If we triple the mass, we triple the rate at which the mass decreases.

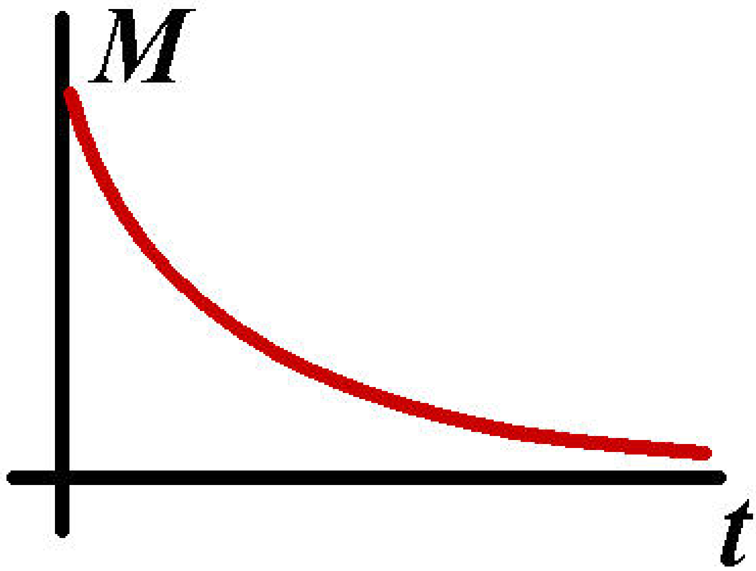


The rate at which the mass decreases is proportional to the mass at any given time.



The rate at which the mass decreases is proportional to the mass at any given time.

$$\frac{dM}{dt} = \lambda M$$

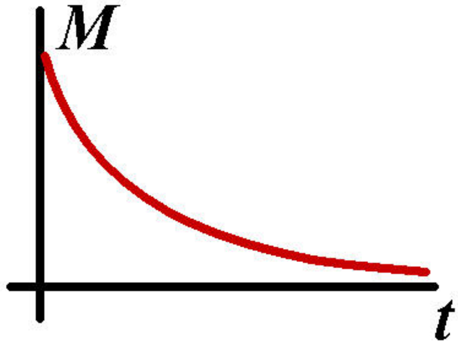


The rate at which the mass decreases is proportional to the mass at any given time.

$$\frac{dM}{dt} = \lambda M \qquad (\lambda \text{ is a negative number})$$

Let $M = M(t)$ be the mass of a radioactive object

$$\frac{dM}{dt} = -\lambda M$$



Population Growth $$\frac{dP}{dt} = kP$$

Bank Account Problem $$\frac{dA}{dt} = 0.01A$$

Radioactive Mass $$\frac{dM}{dt} = -\lambda M$$

$$\frac{dP}{dt} = kP \qquad \frac{dA}{dt} = 0.01A \qquad \frac{dM}{dt} = -\lambda M$$

Find a formula for $y = y(x)$ $\qquad$ $\dfrac{dy}{dx} = ky$

Spring Problem:

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Electrical Circuit Problem:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

How do we solve a differential equation?

$$\frac{dy}{dx} = -2x$$

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$$y = \int -2x\,dx = -x^2 + C$$

$$\frac{dy}{dx} = -2x$$

$$dy = -2x\,dx$$

Separation of Variables

$$\frac{dy}{dx} = -2x$$

$$dy = -2x\,dx$$

$$\int dy = \int -2x\,dx$$

$$y + C_1 = -x^2 + C_2$$

$$\frac{dy}{dx} = -2x$$

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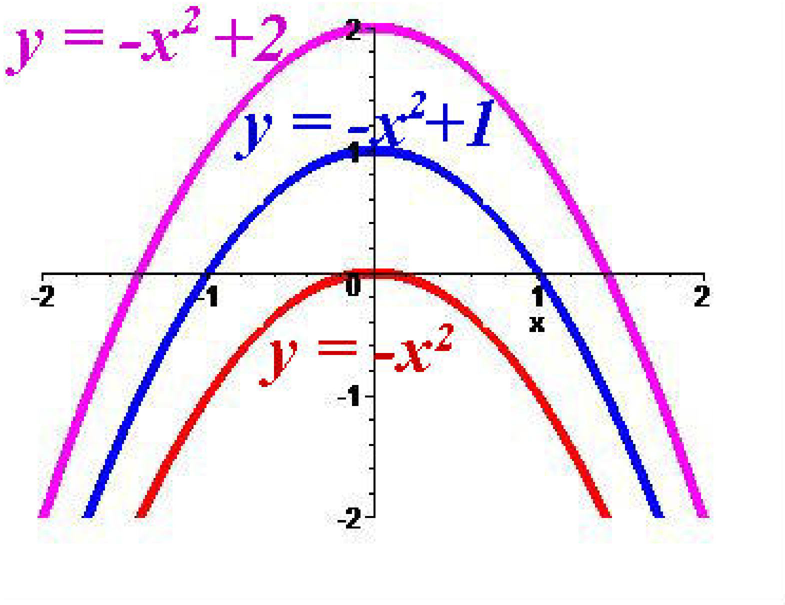
$$y + C_1 = -x^2 + C_2$$

$$y = -x^2 + C$$

(where $C = C_2 - C_1$)

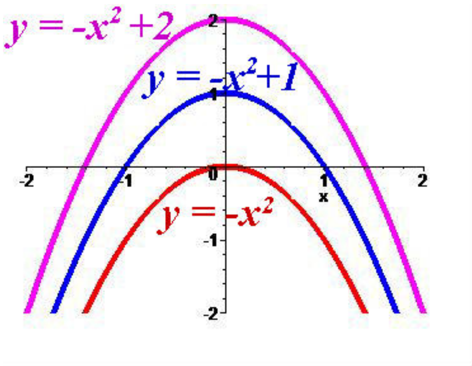
$$y = -x^2 + C$$



$$\frac{dy}{dx} = -2x$$

$$y = -x^2 + C$$

This is called the *general solution*

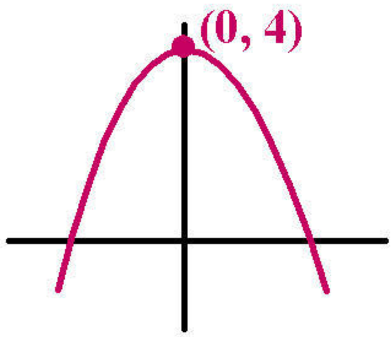


$$\frac{dy}{dx} = -2x \qquad \text{where } y(0) = 4$$

$$y = -x^2 + C$$

If $x = 0$ then $4 = -0^2 + C = C$

$$y = -x^2 + 4 \quad \text{(Particular Solution)}$$



$$\frac{dy}{dx} = -2y \qquad \text{where } y(0) = 1$$

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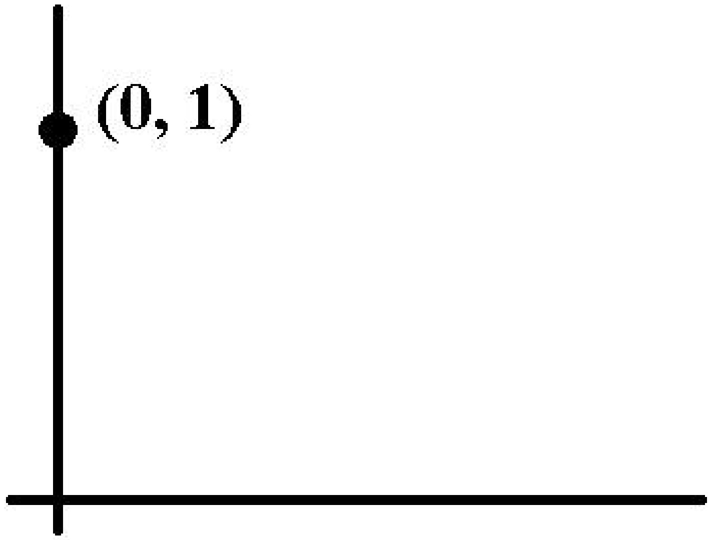
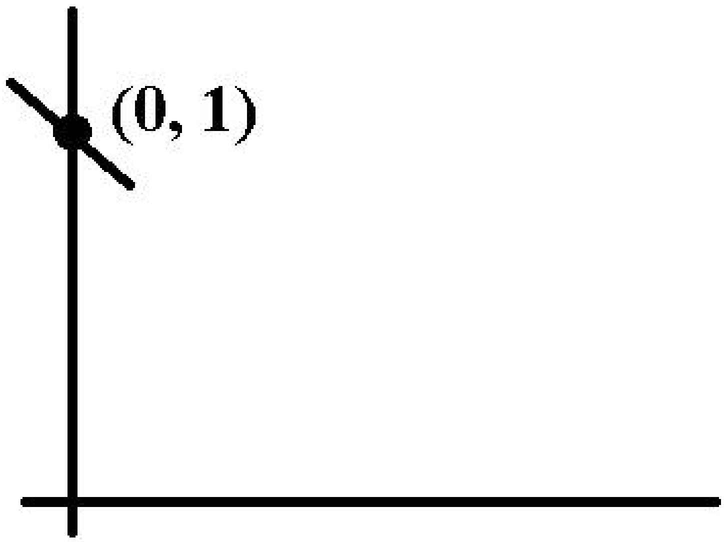
The curve passes through $(0,\ 1)$.



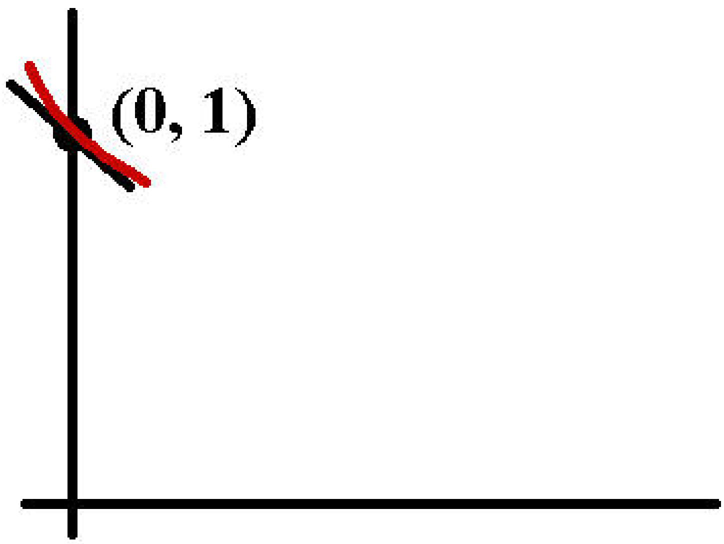
$$\frac{dy}{dx} = -2y \qquad \text{where } y(0) = 1$$

The slope is -2 at $x = 0$



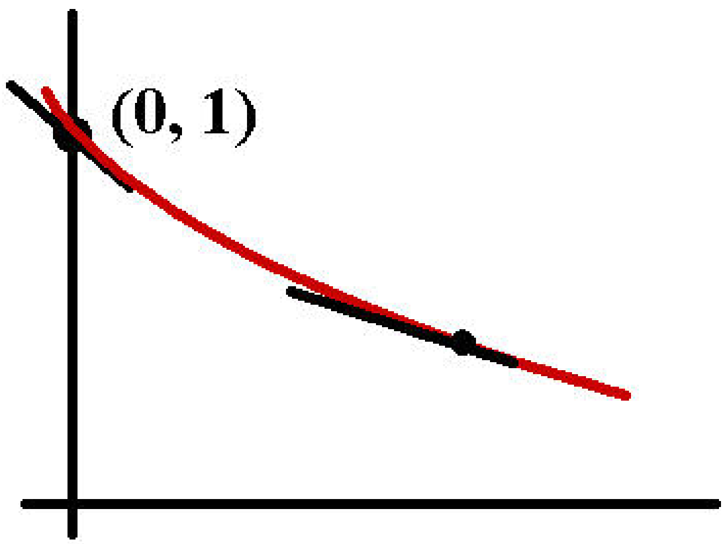
$$\frac{dy}{dx} = -2y \qquad \frac{d^2y}{dx^2} = -2\frac{dy}{dx} = 4y > 0$$

The curve is concave up at $x = 0$



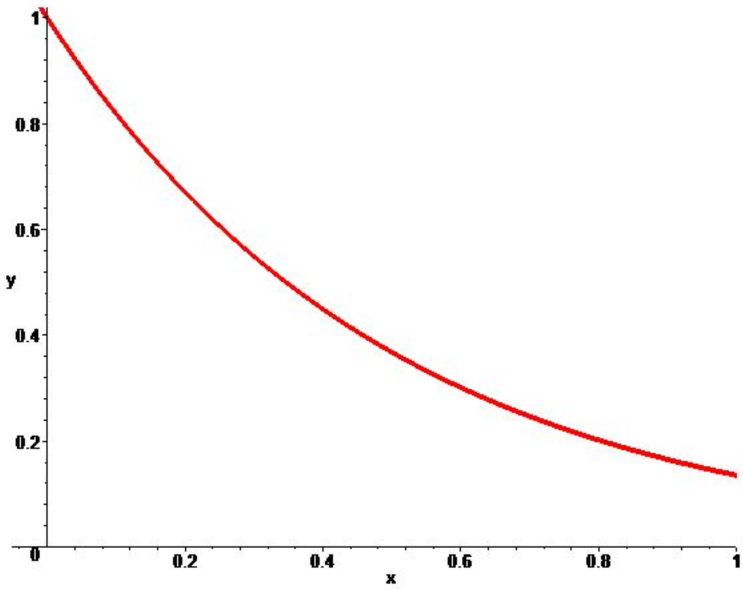
$$\frac{dy}{dx} = -2y$$

As we move to the right, y is getting close to 0 so the slope is getting closer to 0.



$$\frac{dy}{dx} = -2y \qquad \text{where } y(0) = 1$$

The shape of the curve is becoming more evident.



$$\frac{dy}{dx} = -2y \qquad \text{where } y(0) = 1$$

Integrate both sides.

$$y = \int -2y \, dx$$

$$\frac{dy}{dx} = -2y \qquad \text{where } y(0) = 1$$

Integrate both sides.

$$y = \int -2y \, dx = \textbf{???}$$

$\int -2 \sin x \, dx = 2 \cos x + C$

$\int -2e^x \, dx = -2e^x + C$

$\int -2x^3 \, dx = -\frac{1}{2}x^4 + C$

$$\frac{dy}{dx} = -2y \qquad \text{where } y(0) = 1$$

$$dy = -2y\,dx$$

$$\frac{1}{y}\,dy = -2\,dx$$

Separation of variables

$$\frac{dy}{dx} = -2y \qquad \text{where } y(0) = 1$$

$$dy = -2y\,dx$$

$$\frac{1}{y}\,dy = -2\,dx$$

$$\int \frac{1}{y}\,dy = \int -2\,dx$$

$$\ln|y| = -2x + C$$

$$\ln|y| = -2x + C$$

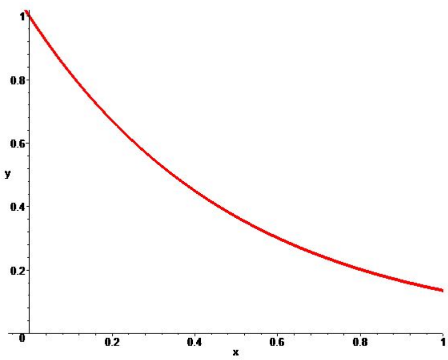
$$|y| = e^{-2x+C}$$

$$y = \pm e^{-2x+C} = \left(\pm e^{C}\right) e^{-2x} = a e^{-2x}$$

$$y = ae^{-2x}$$

Since $y(0) = 1$, we get $1 = ae^0 = a$

$$y = 1e^{-2x}$$



$$\frac{dy}{dx} = 2xy - 2xy^2$$

Find the general solution

$$\frac{dy}{dx} = 2xy - 2xy^2 = 2xy(1-y)$$

$$\frac{1}{y(1-y)}\,dy = 2x\,dx$$

$$\int \frac{1}{y(1-y)}\,dy = \int 2x\,dx$$

$$\frac{dy}{dx} = 2xy - 2xy^2 = 2xy(1-y)$$

$$\frac{1}{y(1-y)}\,dy = 2x\,dx$$

$$\int \frac{1}{y(1-y)}\,dy = \int 2x\,dx$$

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int 2x\,dx$$

$$\ln|y| - \ln|1-y| = x^2 + C$$

$$\ln|y| - \ln|1-y| = x^2 + C$$

$$\ln\left|\frac{y}{1-y}\right| = x^2 + C$$

$$\left|\frac{y}{1-y}\right| = e^{x^2+C} = e^C e^{x^2}$$

$$\frac{y}{1-y} = \pm e^C e^{x^2}$$

$$\frac{y}{1-y} = ae^{x^2}$$

$$\frac{y}{1-y} = ae^{x^2}$$

$$y = ae^{x^2}(1-y)$$

$$y = ae^{x^2} - ae^{x^2}y$$

$$y + ae^{x^2}y = ae^{x^2}$$

$$\left(1 + ae^{x^2}\right)y = ae^{x^2}$$

$$y = \frac{ae^{x^2}}{1 + ae^{x^2}}$$

$$\frac{dy}{dx} = x - 2y$$

Solve.

$$\frac{dy}{dx} = x - 2y$$

Solve. Try separation of variables.

$$dy = (x - 2y)dx$$

$$dy - x\,dx = -2y\,dx$$

$$dy + 2y\,dx = x\,dx$$

$$\frac{1}{x - 2y}dy = dx$$

Now what?