Differential Equations

Some Simple Applications

Population Growth $$\frac{dP}{dt} = kP$$

Bank Account Problem $$\frac{dA}{dt} = 0.01A$$

Radioactive Mass $$\frac{dM}{dt} = -\lambda M$$

All the equations are of the form:

$$\frac{dy}{dt} = ky$$

Solve by separation of variables.

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y} dy = k\,dt$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln y = kt + C$$

$$y = e^{kt+C} = e^C e^{kt} = ae^{kt}$$

The constant a is determined by the initial condition

$$y = ae^{kt}$$

If y_0 denotes $y(0)$ then:

$$y_0 = ae^{k \cdot 0} = ae^0 = a$$

$$y = y_0 e^{kt}$$

Radioactive mass

$$\frac{dM}{dt} = -\lambda M$$
$$M(t) = M_0 e^{-\lambda t}$$

Example: Iodine-131 has a half-life of 8 days. If we start with a 400 grams sample of this substance, how many grams will we have left after 12 days?

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$$M(12) = 400e^{-12\lambda}$$

We need to know λ to complete the calculation

$$M(t) = 400e^{-\lambda t}$$

If the half-life is 8 days, then

$$200 = 400e^{-8\lambda}$$

$$\frac{1}{2} = e^{-8\lambda}$$

$$2 = e^{8\lambda}$$

$$\lambda = \frac{1}{8}\ln 2$$

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If $\lambda = \frac{1}{8}\ln 2$ then:

$$M(12) = 400e^{-12\cdot(1/8)\ln 2} \approx 141 \text{ grams}$$

A truck arrives at a flour storage facility. Hidden in the flour on the truck are 100 flour beetles which then find a home in the flour storage facility.

If $y(t)$ denotes the population of beetles after t days, then:

$$\frac{dy}{dt} = ky$$

$$y = y_0 e^{kt} = 100e^{kt}$$

For flour beetles, the value of k is known to be 0.10

How long will it take for the beetle population to double?

$$y = y_0 e^{kt} = 100e^{kt}$$

How long will it take for the beetle population to double?

We must solve for t so that:

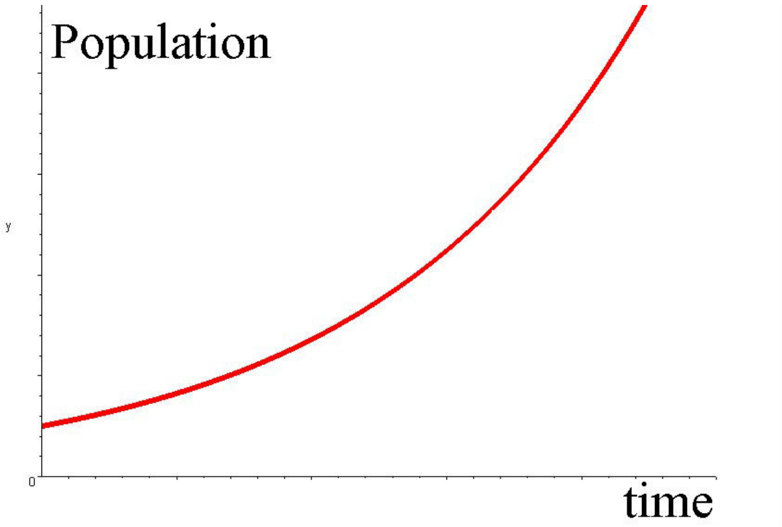
$$200 = 100e^{kt}$$

$$2 = e^{kt}$$

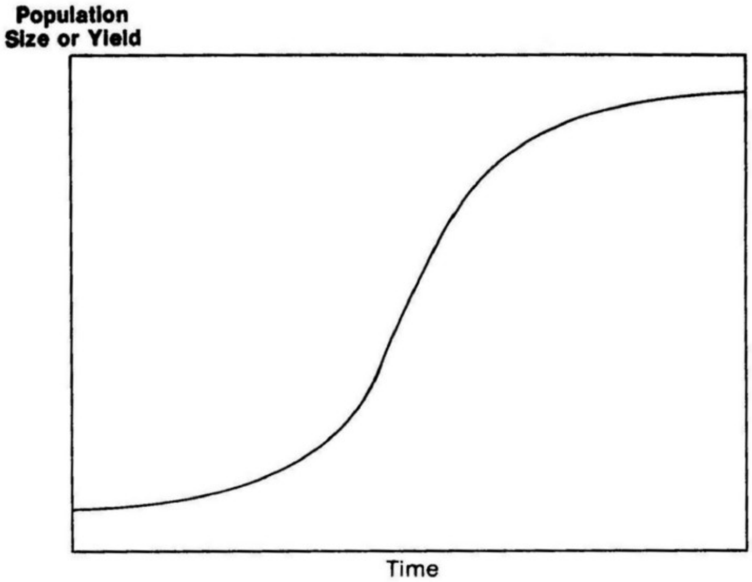
$$\ln 2 = kt$$

$$t = \frac{1}{k}\ln 2 = \frac{1}{0.10}\ln 2 \approx 6.9 \text{ days}$$

$$y = 100e^{kt}$$

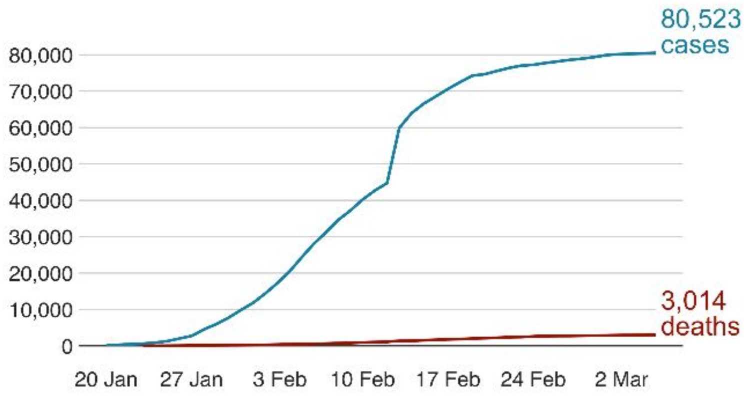


Projected World Population



The Spread of the Corona Virus

Total confirmed cases of coronavirus in the country

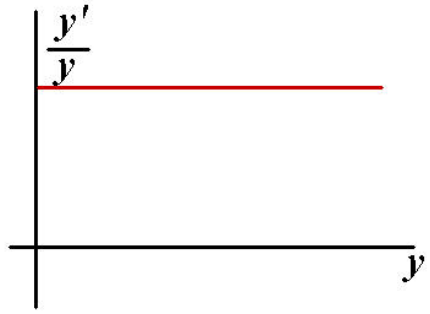


Source: China National Health Commission, WHO, Updated: 5 Mar 06:00 GMT BBC

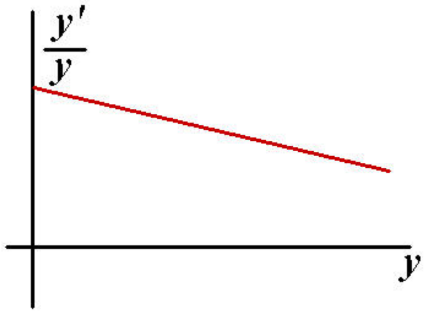
If $\frac{dy}{dt} = ky$ then:

$$\frac{y'}{y} = k$$

Thus, the Malthusian Model assumes that the rate of growth *per individual* remains constant no matter how large the population is.



More realistic model:



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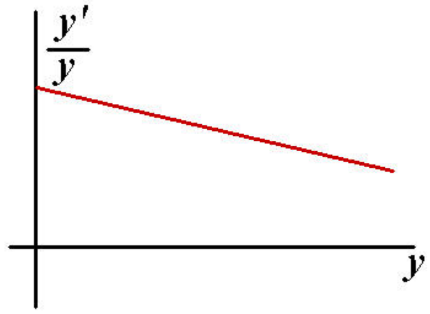
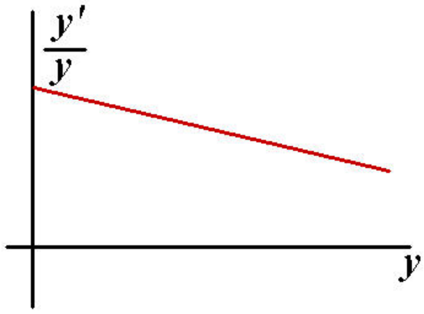


$$\frac{y'}{y} = my + b$$

More realistic model:



$$\frac{y'}{y} = -ay + b$$

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$$\frac{y'}{y} = a\left(-y + \frac{b}{a}\right)$$

Let $L = \frac{b}{a}$

$$\frac{y'}{y} = a\left(L - y\right)$$

$$y' = a(L - y)y$$

This is called the *Logistics Equation*

$$\frac{dy}{dt} = a(L - y)y$$

$$\frac{1}{(L - y)y}\, dy = a\, dt$$

$$\int \frac{1}{(L - y)y}\, dy = \int a\, dt$$

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$$\int \left(\frac{\frac{1}{L}}{y} + \frac{\frac{1}{L}}{L - y} \right) dy = a \int dt$$

$$\frac{1}{L}(\ln|y| - \ln|L - y|) = at + C$$

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$$\ln\left|\frac{y}{L-y}\right| = aLt + LC$$

$$\left|\frac{y}{L-y}\right| = e^{aLt+CL}$$

$$\frac{y}{L-y} = \pm e^{aLt+CL} = \left(\pm e^{CL}\right) e^{aLt} = ke^{alt}$$

$$\frac{y}{L-y} = ke^{aLt}$$

$$y = \left(ke^{aLt}\right)(L-y)$$

$$y = kLe^{aLt} - kye^{aLt}$$

$$y + kye^{aLt} = kLe^{aLt}$$

$$\left(1 + ke^{aLt}\right) y = kLe^{aLt}$$

$$y = \frac{kLe^{aLt}}{1 + ke^{aLt}}$$

$$y = \frac{kLe^{aLt}}{1 + ke^{aLt}} = \frac{kL}{e^{-aLt} + k}$$

$$\lim_{t \to \infty} \frac{kL}{e^{-aLt} + k} = \frac{kL}{0 + k} = L$$