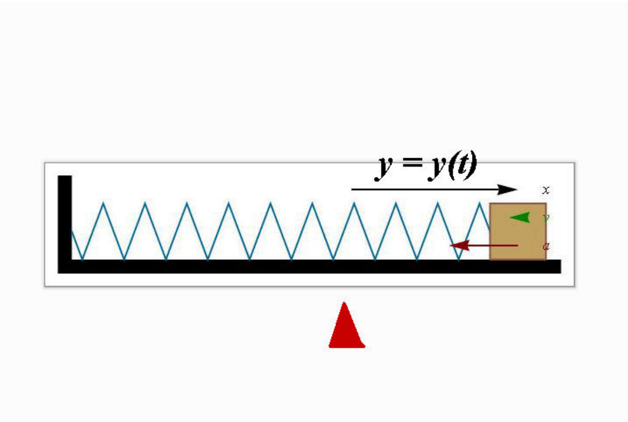
Differential Equations Review
Dr. E. Jacobs
Today's Topic : Second Order Equations

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$



Heat Equation:

Find a function $u = u(x, t)$ that solves:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Find the function $y = y(t)$ that solves:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

$$\frac{d^2y}{dt^2} - 4y = 0$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0$$

$$\frac{d^3y}{dt^3} - \frac{dy}{dt} = 0$$

Let's start simply:

$$\frac{dy}{dt} - 4y = 0$$

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$$\frac{dy}{dt} = 4y$$

$$\frac{1}{y}\,dy = 4\,dt$$

$$\frac{dy}{dt} - 4y = 0$$

$$\frac{dy}{dt} = 4y$$

$$\int \frac{1}{y}\,dy = \int 4\,dt$$

$$\ln y = 4t + C$$

$$y = e^{4t+C} = ae^{4t}$$

where $a = e^C$

The general solution of $\frac{dy}{dt} - 4y = 0$ is ae^{4t}

Next, solve

$$\frac{d^2y}{dt^2} - 4y = 0$$

The general solution of $\frac{dy}{dt} - 4y = 0$ is ae^{4t}

Next, solve

$$\frac{d^2y}{dt^2} - 4y = 0$$

Will we still get solutions of the form e^{rt} ?

$$\frac{d^2y}{dt^2} - 4y = 0$$

If $y = e^{rt}$ then $\frac{dy}{dt} = re^{rt}$ and $\frac{d^2y}{dt^2} = r^2e^{rt}$

Substitute into the differential equation:

$$r^2e^{rt} - 4e^{rt} = 0$$

$$\frac{d^2y}{dt^2} - 4y = 0$$

If $y = e^{rt}$ then $\frac{dy}{dt} = re^{rt}$ and $\frac{d^2y}{dt^2} = r^2e^{rt}$

Substitute into the differential equation:

$$r^2e^{rt} - 4e^{rt} = 0$$

Divide both sides by e^{rt}

$$r^2 - 4 = 0 \quad \text{and therefore} \quad r = \pm 2$$

The equation $\frac{d^2y}{dt^2} - 4y = 0$ has two solutions:

$$y_1 = e^{2t} \quad \text{and} \quad y_2 = e^{-2t}$$

Are there any other solutions?

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Are there any other solutions?

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{2t} + c_2 e^{-2t}$$

A linear combination of y_1 and y_2

The equation $\frac{d^2y}{dt^2} - 4y = 0$ has two solutions:

$$y_1 = e^{2t} \quad \text{and} \quad y_2 = e^{-2t}$$

Are there any other solutions?

$$y = c_1 e^{2t} + c_2 e^{-2t} \qquad \frac{d^2y}{dt^2} = 4c_1 e^{2t} + 4c_2 e^{-2t}$$

$$\begin{aligned}\frac{d^2y}{dt^2} - 4y &= \left(4c_1 e^{2t} + 4c_2 e^{-2t}\right) - 4\left(c_1 e^{2t} + c_2 e^{-2t}\right) \\ &= 0\end{aligned}$$

$$\frac{d^2y}{dt^2} - 4y = 0$$

$$y = c_1e^{2t} + c_2e^{-2t}$$

This is the general solution.

$$\frac{d^2y}{dt^2} - 4y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 2$$

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

$$\frac{d^2y}{dt^2} - 4y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 2$$

$$y = c_1e^{2t} + c_2e^{-2t}$$

If $y(0) = 0$ then $0 = c_1 + c_2$ so $c_2 = -c_1$

$$y = c_1e^{2t} - c_1e^{-2t}$$

$$\frac{d^2y}{dt^2} - 4y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 2$$

$$y = c_1e^{2t} - c_1e^{-2t}$$

$$y' = 2c_1e^{2t} + 2c_1e^{-2t}$$

$2 = y'(0)$ implies that $2 = 2c_1 + 2c_1$

$$y = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t}$$

The Hyperbolic Functions

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\frac{d^2y}{dt^2} - 4y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 2$$

$$y = c_1 e^{2t} - c_1 e^{-2t}$$

$$y' = 2c_1 e^{2t} + 2c_1 e^{-2t}$$

$2 = y'(0)$ implies that $2 = 2c_1 + 2c_1$

$$y = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t} = \sinh 2t$$

Find $y = y(x)$ that solve:

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$$

Start by looking for solutions of the form e^{rx}

Find $y = y(x)$ that solve:

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$$

Start by looking for solutions of the form e^{rx}

$$r^3e^{rx} - 2r^2e^{rx} - 2re^{rx} = 0$$

$$r^3 - 2r^2 - 2r = 0$$

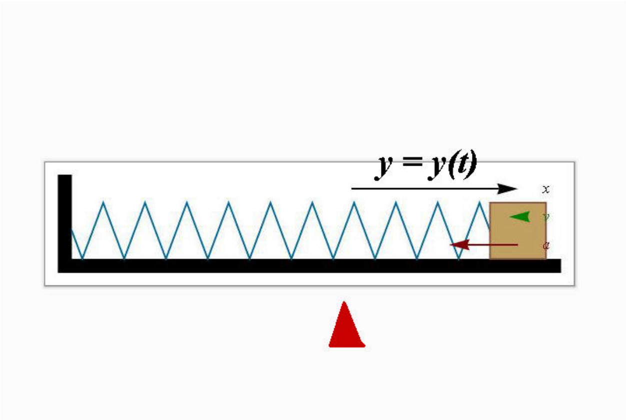
$$r^3 - 2r^2 - 2r = 0$$

$$r\left(r^2 - 2r - 2\right) = 0$$

$r = 0$ is one solution.
For the others, solve $r^2 - 2r - 2 = 0$ to get $r = 1 \pm \sqrt{3}$

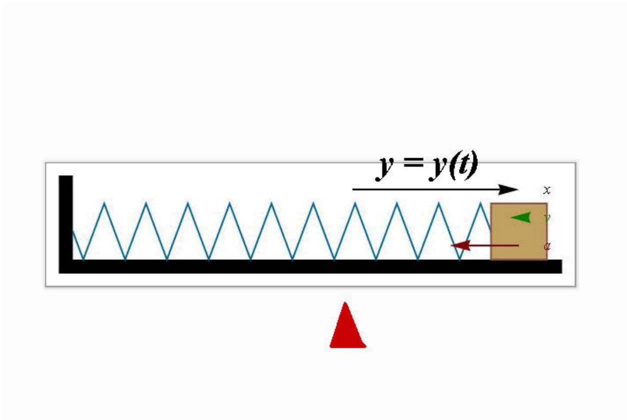
$$y = c_1 e^{0x} + c_2 e^{(1+\sqrt{3})x} + c_3 e^{(1-\sqrt{3})x}$$

Spring Motion:



Hooke's Law: The force of the spring is proportional to the displacement of the object.

$$F_s = -ky$$



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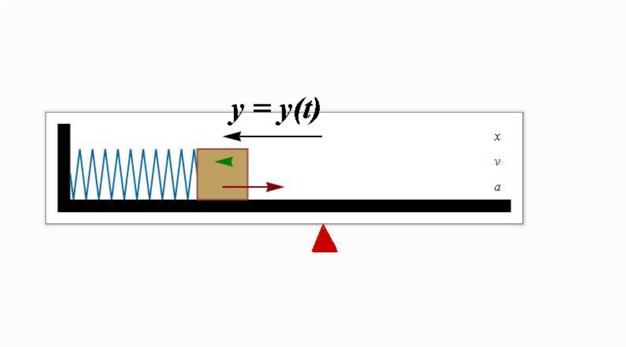
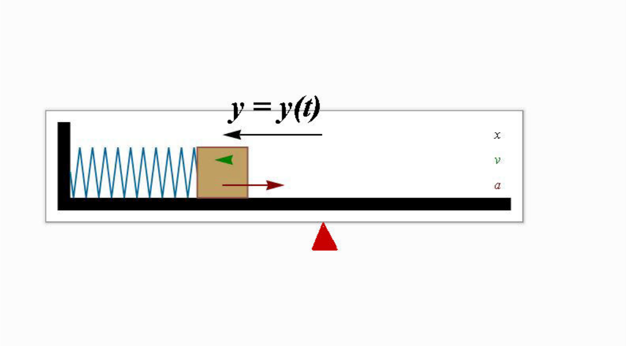


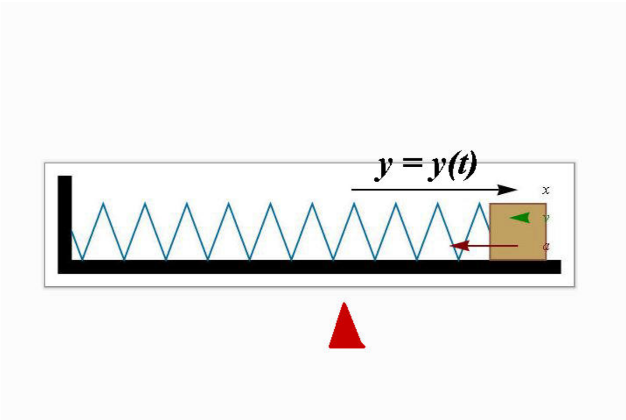
Resistive (damping) force is proportional to the velocity of the object

$$F_r = -\beta v$$



$$(\text{mass})(\text{acceleration}) = \text{Net force}$$

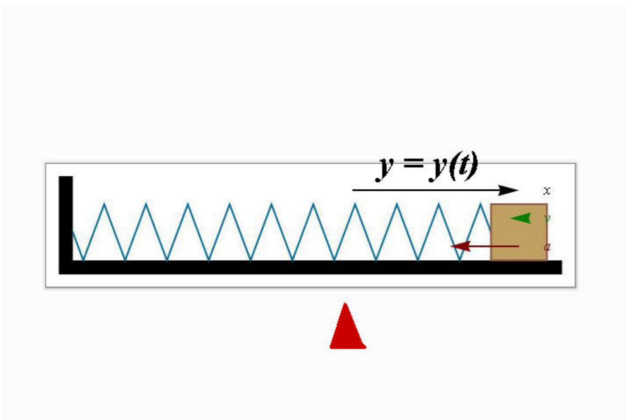
$$ma = F_s + F_r$$



$$ma = F_s + F_r$$

$$m\frac{d^2y}{dt^2} = -ky - \beta\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 5$ and $k = 4$.

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0$$

Find the general solution

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0$$

Look for solutions of the form e^{rt}

$$r^2e^{rt} + 5re^{rt} + 4e^{rt} = 0$$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0$$

Look for solutions of the form e^{rt}

$$r^2e^{rt} + 5re^{rt} + 4e^{rt} = 0$$

$$r^2 + 5r + 4 = 0$$

$$(r+1)(r+4) = 0$$

$$r = -1 \quad r = -4$$

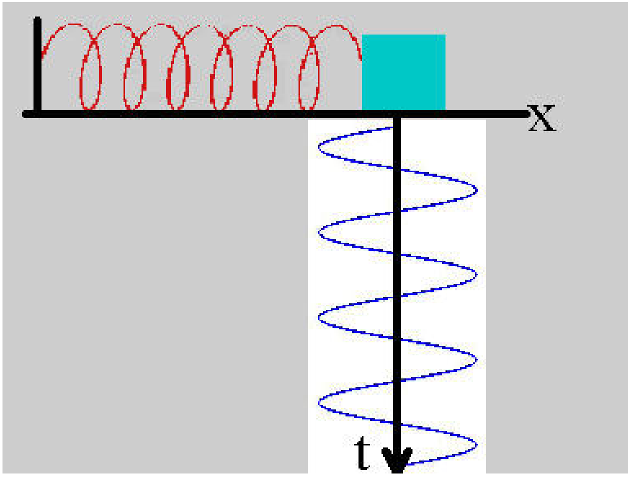
$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0$$

$y = e^{rt}$ is a solution if $r = -1$ or $r = -4$.

$$y_1 = e^{-1t} \qquad y_2 = e^{-4t}$$

The general solution is:

$$y = c_1 e^{-t} + c_2 e^{-4t}$$


x
t

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 0$ and $k = 4$.

$$\frac{d^2y}{dt^2} + 4y = 0$$

Find the general solution

$$\frac{d^2y}{dt^2} + 4y = 0$$

Substitute $y = e^{rt}$

$$r^2 + 4 = 0$$

$$r^2 = -4$$

What does e^{rt} mean if r is an imaginary number?