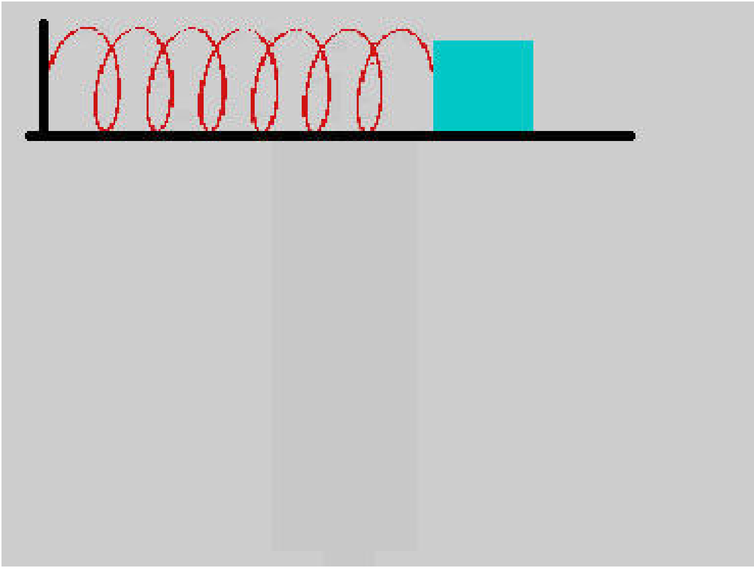
Differential Equations
Dr. E. Jacobs

Today's Topic : Imaginary Numbers

Review:

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

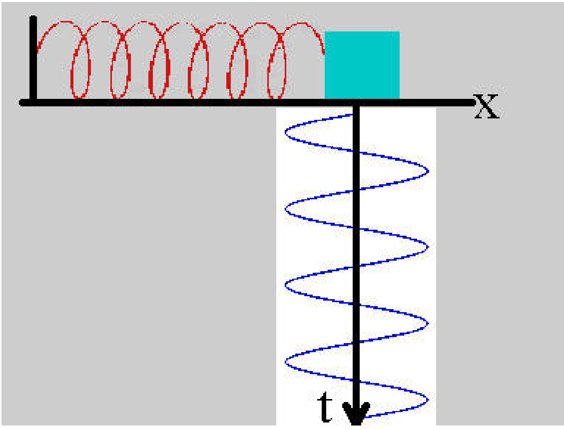


$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 0$ and $k = 4$.

$$\frac{d^2y}{dt^2} + 4y = 0$$

$$\frac{d^2y}{dt^2} + 4y = 0$$



Substitute $y = e^{rt}$ into $y'' + 4y = 0$

$$r^2 + 4 = 0$$

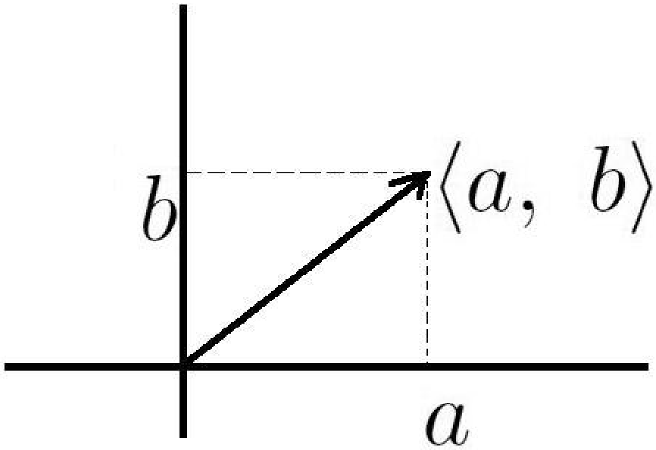
$$r^2 = -4$$

$$r = \pm\sqrt{-4} = \pm 2\sqrt{-1} = \pm 2i$$

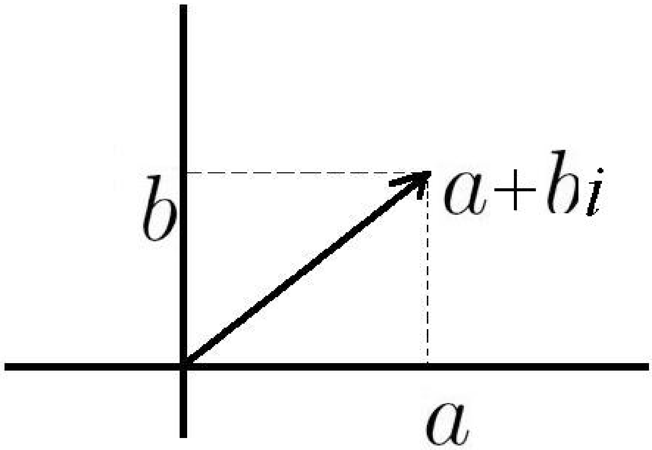
What do e^{2it} and e^{-2it} mean?

$i = \sqrt{-1}$ is called an imaginary number

Suppose the coordinates of a vector are a and b

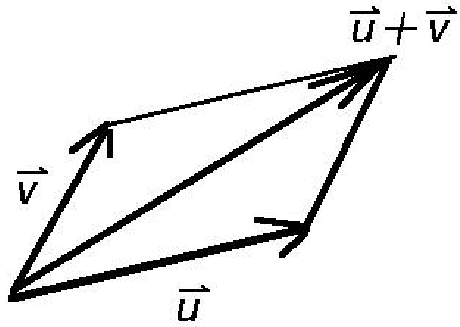


Regard $a + bi$ as an alternative notation for a vector



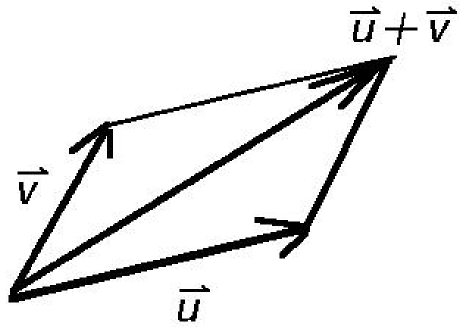
$$\mathbf{u} = \langle a,\ b \rangle \qquad \mathbf{v} = \langle c,\ d \rangle$$

$$\mathbf{u} + \mathbf{v} = \langle a + c,\ b + d \rangle$$



$$\mathbf{u} = a + bi \qquad \mathbf{v} = c + di$$

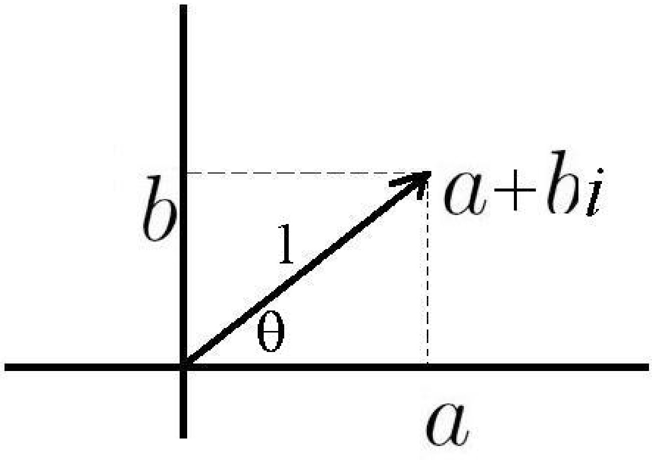
$$\mathbf{u} + \mathbf{v} = a + c + (b + d)i$$



What does complex multiplication mean?

$$
\begin{aligned}
(1+i)(2+i) &= 2 + 1i + 2i + i^2 \\
&= 2 + 3i - 1 \\
&= 1 + 3i
\end{aligned}
$$

Suppose the vector is a unit vector



$$a = \cos\theta \qquad b = \sin\theta$$

$$a + bi = \cos\theta + i\sin\theta$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

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$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$i^2 = -1$$

$$i^3 = i^2 i = -i$$

$$i^4 = i^2 i^2 = (-1)(-1) = 1$$

$$i^5 = i^4 i^1 = 1i$$

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots
\end{aligned}
$$

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdot \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdot\right.
\end{aligned}
$$

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\Bigg(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\
&= \cos\theta + i\sin\theta
\end{aligned}
$$

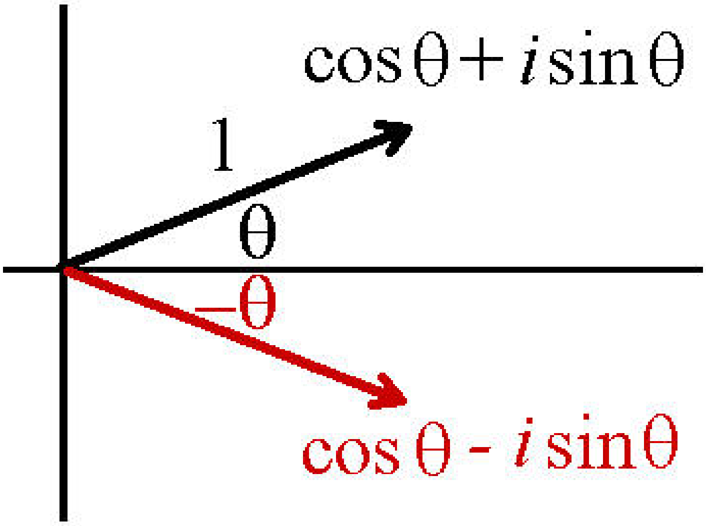
Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Replace θ with $-\theta$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$



$$\text{Substitute } y = e^{rt} \text{ into } \frac{d^2y}{dt^2} + 4y = 0$$

$$r^2 + 4 = 0 \quad \text{which implies that} \quad r = \pm 2i$$

$$y = c_1 e^{2it} + c_2 e^{-2it}$$

This answer can be written in terms of $\sin 2t$ and $\cos 2t$