Differential Equations
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Today's Topic : The Complex Root Case

Review:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

Simplify:

$$e^{a+bi}$$

$$
\begin{aligned}
e^{a+bi} &= e^a e^{bi} \\
&= e^a(\cos b + i \sin b)
\end{aligned}
$$

Find $y = y(x)$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 20y = 0$$

$$(D^2 + 4D + 20)y = 0$$

Substitute e^{rx}

$$r^2 + 4r + 20 = 0$$

$$r^2 + 4r + 20 = 0$$

$$r^2 + 4r + 4 = -16$$

$$(r + 2)^2 = -16$$

$$r + 2 = \pm 4i$$

$$r = -2 \pm 4i$$

$$(D^2 + 4D + 20)y = 0$$

Substitute e^{rx}

$$r = -2 \pm 4i$$

$$\begin{aligned} y &= c_1 e^{(-2+4i)x} + c_2 e^{(-2-4i)x} \\ &= e^{-2x}\left(c_1 e^{4ix} + c_2 e^{-4ix}\right) \end{aligned}$$

$$
\begin{aligned}
y &= c_1 e^{(-2+4i)x} + c_2 e^{(-2-4i)x} \\
&= e^{-2x}\left(c_1 e^{4ix} + c_2 e^{-4ix}\right) \\
&= e^{-2x}(c_1(\cos 4x + i\sin 4x) + c_2(\cos 4x - i\sin 4x)) \\
&= e^{-2x}((c_1 + c_2)\cos 4x + (c_1 i - c_2 i)\sin 4x)
\end{aligned}
$$

Let $A = c_1 + c_2$ and $B = c_1 i - c_2 i$

$$
\begin{aligned}
y &= c_1 e^{(-2+4i)x} + c_2 e^{(-2-4i)x} \\
&= e^{-2x}\left(c_1 e^{4ix} + c_2 e^{-4ix}\right) \\
&= e^{-2x}(c_1(\cos 4x + i \sin 4x) + c_2(\cos 4x - i \sin 4x)) \\
&= e^{-2x}((c_1 + c_2)\cos 4x + (c_1 i - c_2 i)\sin 4x) \\
&= A e^{-2x}\cos 4x + B e^{-2x}\sin 4x
\end{aligned}
$$

If $\alpha \pm \beta i$ solves the characteristic equation, then the solution of the differential equation will be a linear combination of:

$$e^{\alpha x} \cos \beta x \quad \text{and} \quad e^{\alpha x} \sin \beta x$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

Suppose $L = 1$, $R = 4$ and $C = 0.05$

$$\frac{d^2Q}{dt^2} + 4\frac{dQ}{dt} + 20Q = 0$$

$$Q(t) = Ae^{-2t}\cos 4t + Be^{-2t}\sin 4t$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 4$ and $k = 20$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 0$$

$$y(t) = Ae^{-2t}\cos 4t + Be^{-2t}\sin 4t$$

$$y(t) = Ae^{-2t}\cos 4t + Be^{-2t}\sin 4t$$

Initial Conditions: $y(0) = 0$ and $y'(0) = 8$

$$0 = A \cdot 1 + B \cdot 0 = A$$

$$y(t) = Be^{-2t}\sin 4t$$

$$y(t) = Be^{-2t} \sin 4t$$

$$y'(t) = 4Be^{-2t} \cos 4t - 2Be^{-2t} \sin 4t$$

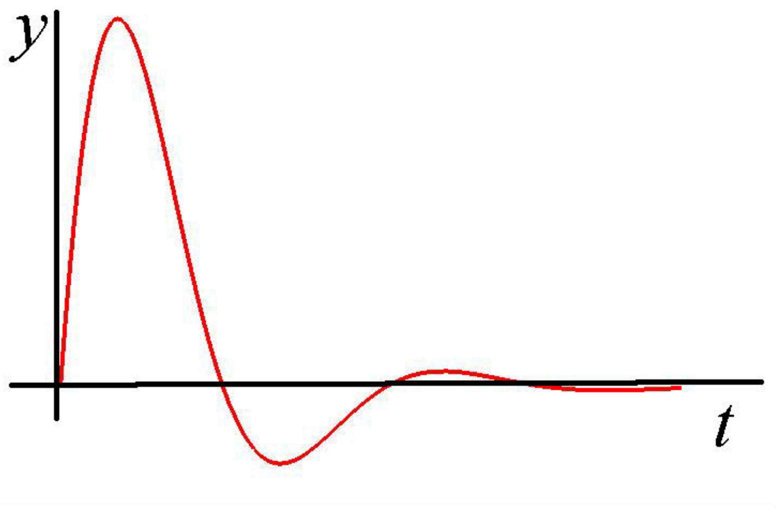
$$y'(0) = 4B \cdot 1 - 2B \cdot 0 = 4B$$

$$8 = 4B$$

$$B = 2$$

$$y(t) = 2e^{-2t} \sin 4t$$

$$y(t) = 2e^{-2t}\sin 4t$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$