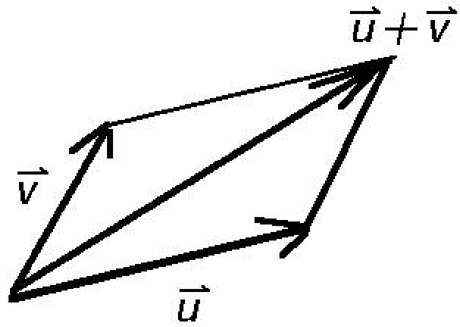
Differential Equations
Dr. E. Jacobs

Today's Topic : $re^{i\theta}$

$$\mathbf{u} = a + bi \qquad \mathbf{v} = c + di$$

$$\mathbf{u} + \mathbf{v} = a + c + (b + d)i$$

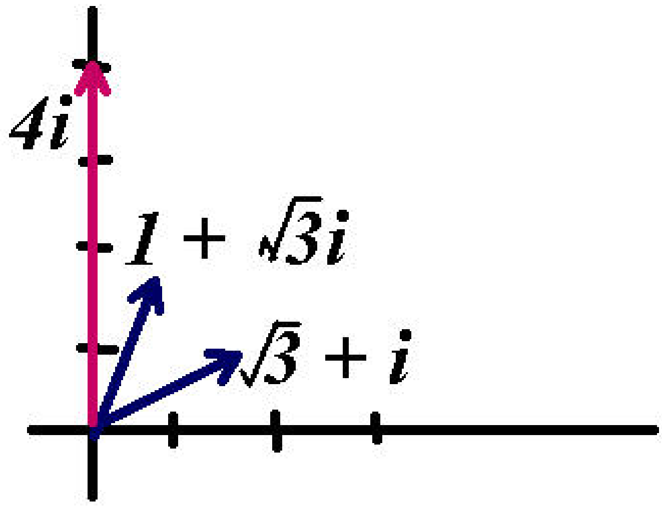


Multiplication of complex numbers

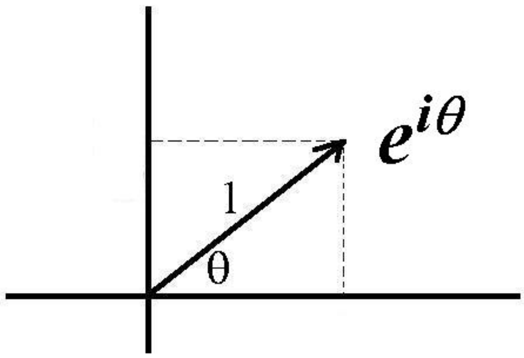
$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + (bi)(di) \\ &= ac - bd + (ad + bc)i\end{aligned}$$

$$
\begin{aligned}
(\sqrt{3}+i)(1+i\sqrt{3}) &= \sqrt{3}+1i+3i+i^2\sqrt{3} \\
&= \sqrt{3}+4i-\sqrt{3} \\
&= 4i
\end{aligned}
$$

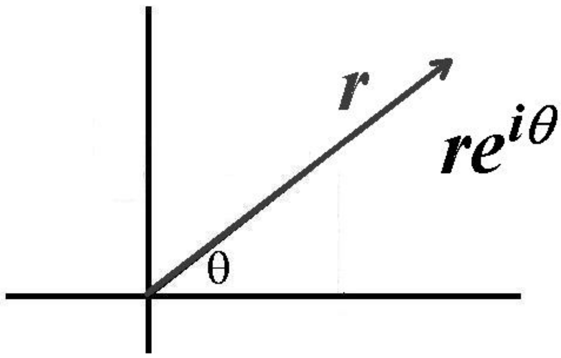
$$(\sqrt{3}+i)(1+i\sqrt{3}) = 4i$$



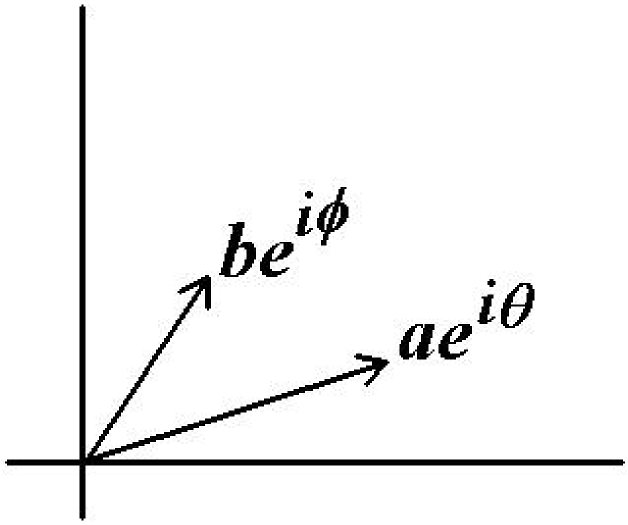
$$e^{i\theta} = \cos\theta + i\sin\theta$$



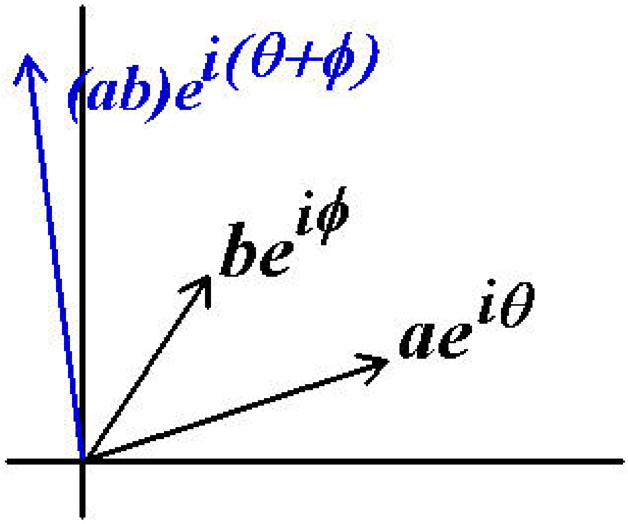
$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$



$ae^{i\theta}$ is a vector of length a at angle θ
$be^{i\phi}$ is a vector of length b at angle ϕ



$$ae^{i\theta} \cdot be^{i\phi} = (ab)e^{i(\theta+\phi)}$$



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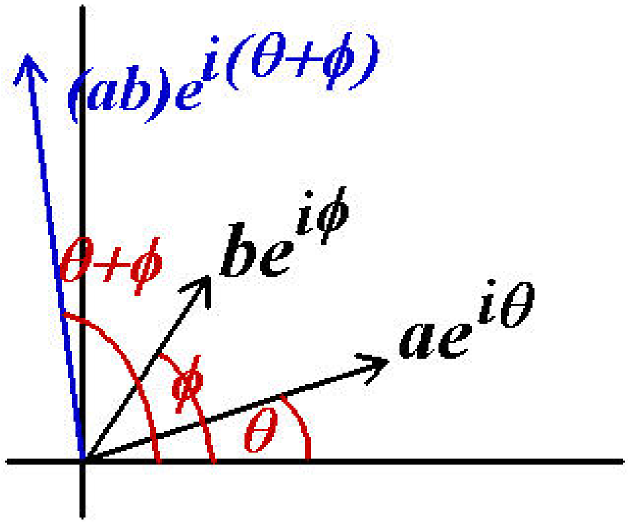
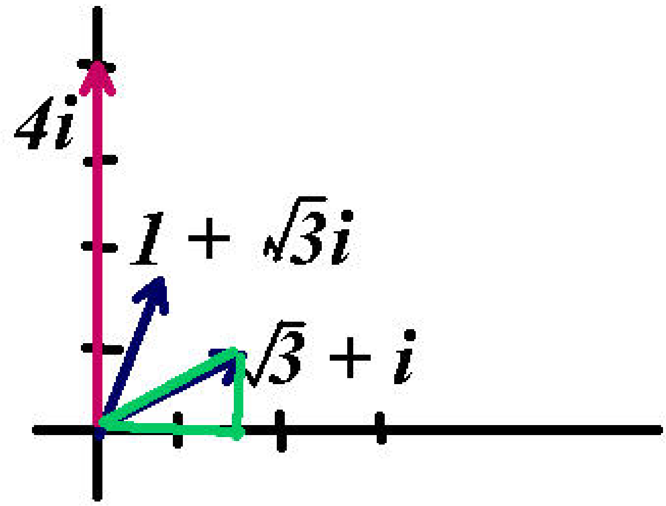
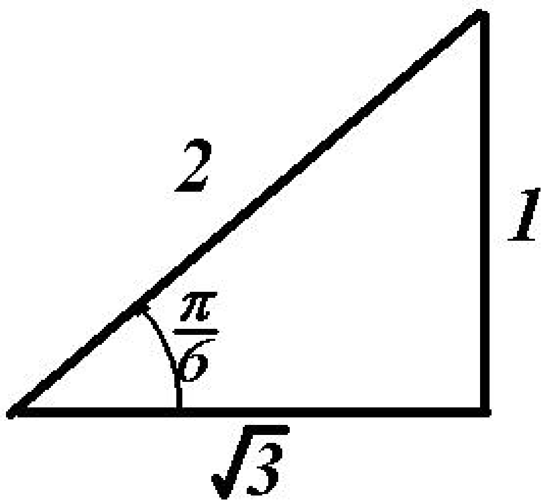


$$\mathbf{u} = \sqrt{3} + 1i$$



$$\mathbf{u} = \sqrt{3} + 1i = 2e^{i\frac{\pi}{6}}$$



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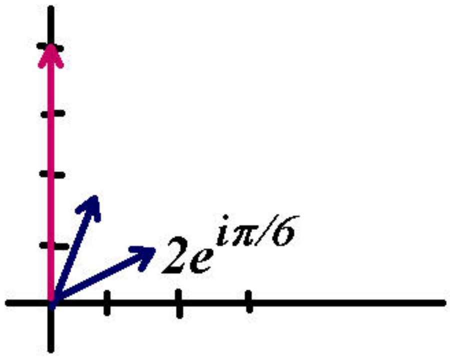
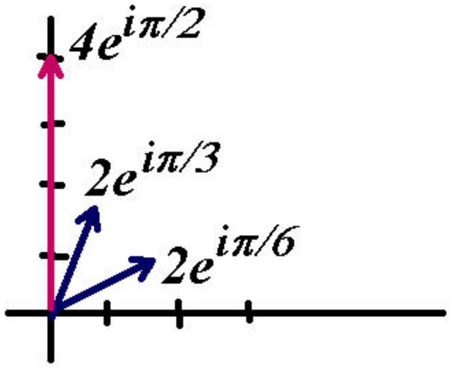
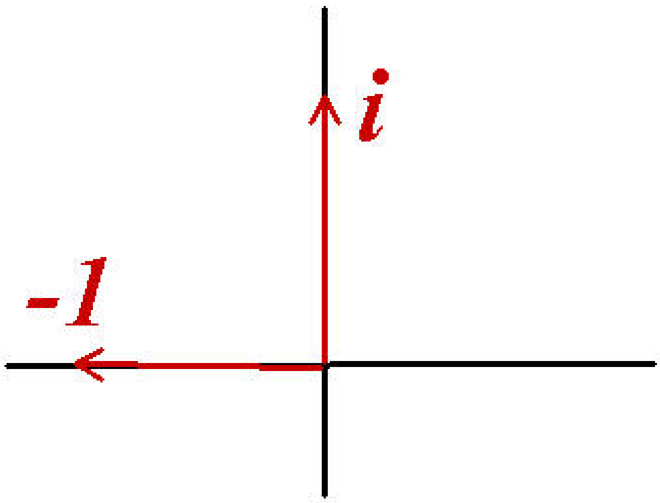


$$\mathbf{u} = \sqrt{3} + 1i = 2e^{i\frac{\pi}{6}} \qquad \mathbf{v} = 1 + i\sqrt{3} = 2e^{i\frac{\pi}{3}}$$



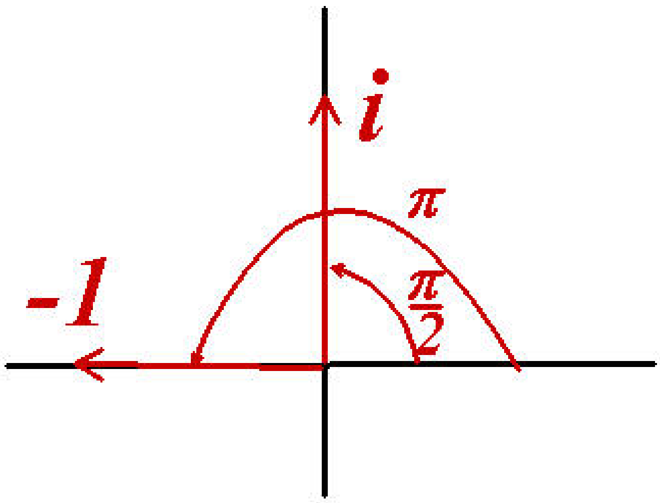
$$i^2 = -1$$

$$i \cdot i = -1$$



$$i \cdot i = -1$$

$$e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{2}} = e^{\pi i}$$



$$e^{\pi i} = -1$$

$$e^{\pi i} + 1 = 0$$

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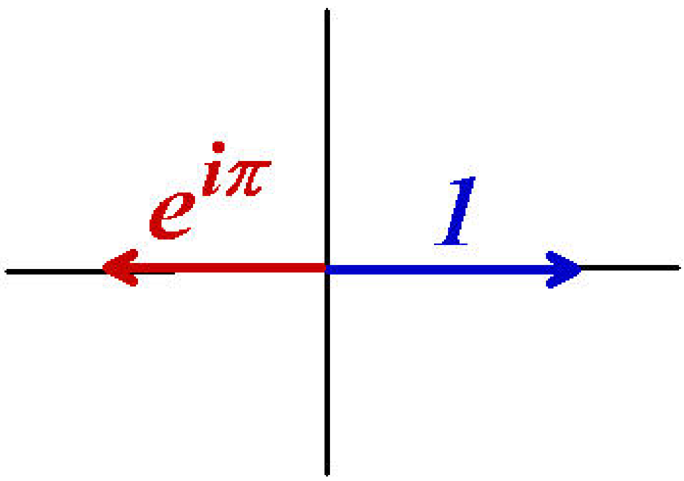
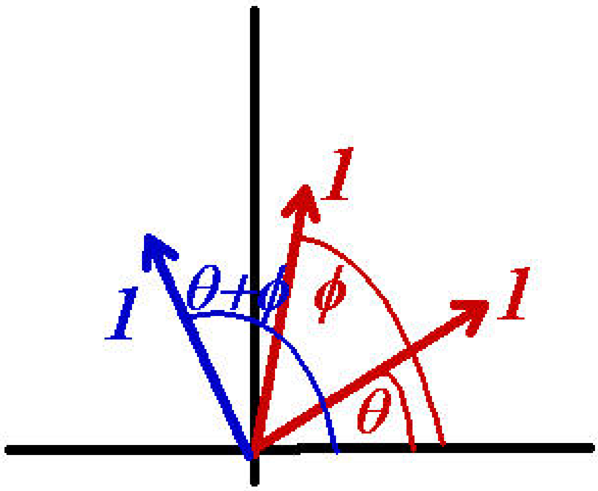


$$e^{i(\theta+\phi)} = e^{i\theta} \cdot e^{i\phi}$$



$$e^{i\theta} \cdot e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$
$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$$

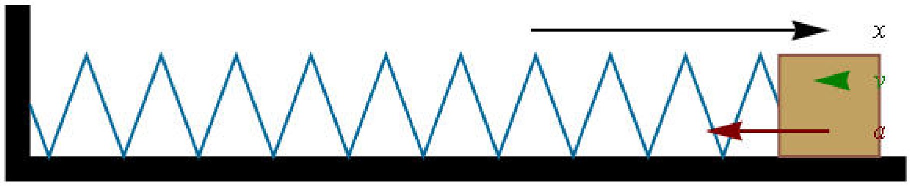
$$e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$e^{i(\theta+\phi)} = e^{i\theta} \cdot e^{i\phi}$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

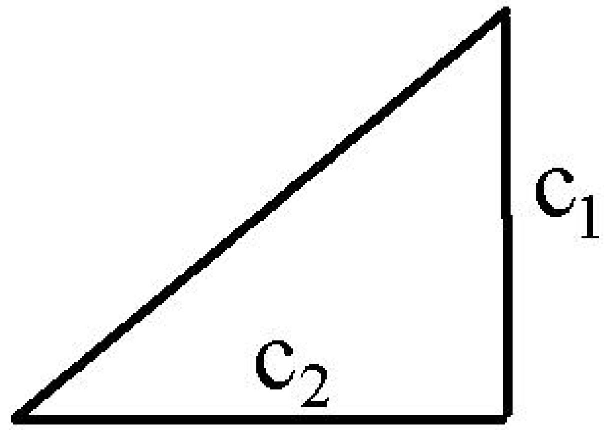
$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$



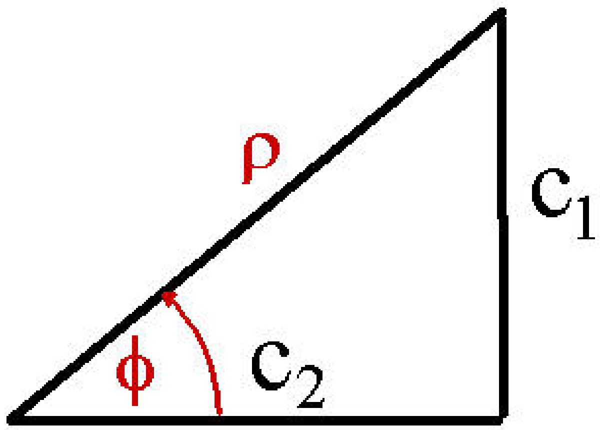
Underdamped Case:

$$y(t) = c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$

Construct a triangle with sides c_1 and c_2



$$\cos\phi = \frac{c_2}{\rho} \qquad \sin\phi = \frac{c_1}{\rho}$$
$$c_2 = \rho\cos\phi \qquad c_1 = \rho\sin\phi$$



$$
\begin{aligned}
y(t) &= e^{-\frac{\beta t}{2m}}(c_1 \cos \omega t + c_2 \sin \omega t) \\
&= e^{-\frac{\beta t}{2m}}(\rho \sin \phi \cos \omega t + \rho \cos \phi \sin \omega t) \\
&= \rho e^{-\frac{\beta t}{2m}}(\sin \phi \cos \omega t + \cos \phi \sin \omega t) \\
&= \rho e^{-\frac{\beta t}{2m}} \sin(\omega t + \phi)
\end{aligned}
$$

$$y(t) = \rho e^{-\frac{\beta t}{2m}} \sin(\omega t + \phi)$$