Preparation for Fourier Series
Trigonometric Integrals

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Power Series:

$$f(x) = \sum_{n} c_n x^n$$

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Fourier Series:

$$f(x) = \sum_n (a_n \cos nx + b_n \sin nx)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

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$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx$$

If $n = m = 0$ then:

$$\int_{-\pi}^{\pi} \sin 0x \sin 0x \, dx = \int_{-\pi}^{\pi} 0 \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos 0x \cos 0x \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

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$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

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$$\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$$

$$\cos(\theta + \phi) + \cos(\theta - \phi) = 2\cos\theta \cos\phi$$

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$$\cos(\theta - \phi) - \cos(\theta + \phi) = 2\sin\theta\sin\phi$$

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Let $\theta = nx$ and $\phi = mx$

$$\cos((n + m)x) + \cos((n - m)x) = 2\cos nx\cos mx$$

$$\cos((n - m)x) - \cos((n + m)x) = 2\sin nx\sin mx$$

$$\cos nx \cos mx = \frac{1}{2}(\cos((n+m)x) + \cos((n-m)x))$$

$$\sin nx \sin mx = \frac{1}{2}(\cos((n-m)x) - \cos((n+m)x))$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n-m)x) - \cos((n+m)x)) \, dx$$

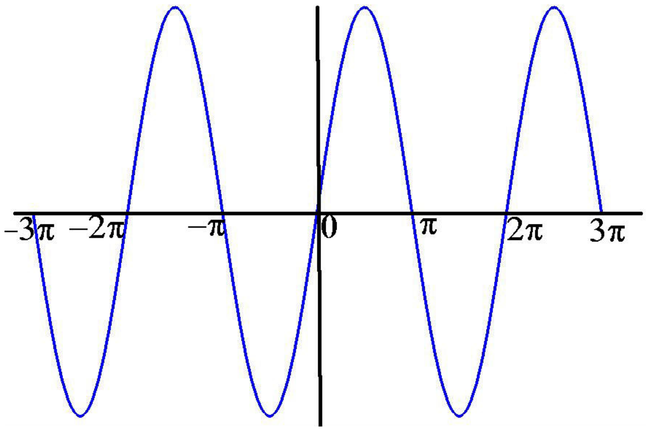
We already know that this integral is 0 if $n = m = 0$ so just consider n and m to be positive integers

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n-m)x) - \cos((n+m)x)) \, dx$$

$$= \frac{1}{2} \left[\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m} \right]_{-\pi}^{\pi}$$

$\sin(-3\pi) = 0 \qquad \sin(-2\pi) = 0 \qquad \sin(-1\pi) = 0$

$\sin(3\pi) = 0 \qquad \sin(2\pi) = 0 \qquad \sin(1\pi) = 0$



$$
\begin{aligned}
\int_{-\pi}^{\pi} \sin nx \sin mx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n{-}m)x) {-} \cos((n{+}m)x)) \, dx \\
&= \frac{1}{2} \left[\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m} \right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}
$$

(Assuming $n \neq m$)

If $n = m$ then:

$$\int_{-\pi}^{\pi} \sin nx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n-n)x) - \cos((n+n)x)) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx) \, dx$$

$$= \frac{1}{2} \left[x - \frac{\sin(2nx)}{2n} \right]_{-\pi}^{\pi}$$

$$= \pi$$

If n and m are positive integers then:

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n+m)x) + \cos((n-m)x)) \, dx$$

We already know that this integral is 2π if $n = m = 0$ so just consider n and m to be positive integers

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n+m)x) + \cos((n-m)x)) \, dx$$

$$= \frac{1}{2} \left[\frac{\sin((n+m)x)}{n+m} + \frac{\sin((n-m)x)}{n-m} \right]_{-\pi}^{\pi}$$

$$= 0$$

(Assuming $n \neq m$)

If $n = m$ then:

$$\int_{-\pi}^{\pi} \cos nx \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n+n)x) + \cos((n-n)x)) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (\cos 2nx + 1) \, dx$$

$$= \frac{1}{2} \left[\frac{\sin(2nx)}{2n} + x \right]_{-\pi}^{\pi}$$

$$= \pi$$

If n and m are positive integers then:

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

If $n = m = 0$ then:

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$