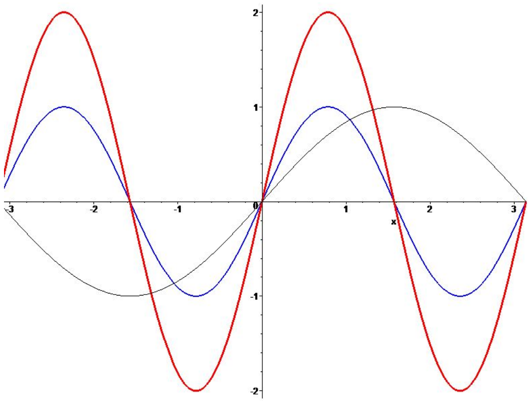
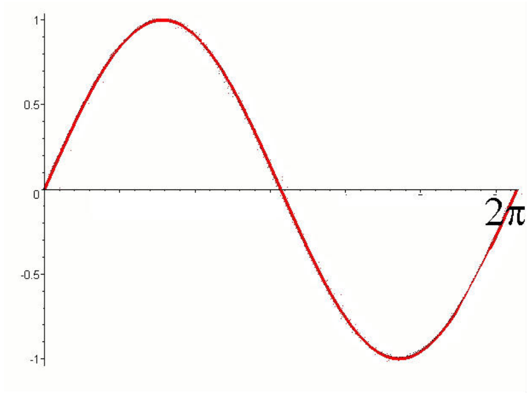
Introduction to Fourier Series

Dr. Elliott Jacobs

$$f(x) = \sum_n (a_n \cos nx + b_n \sin x)$$

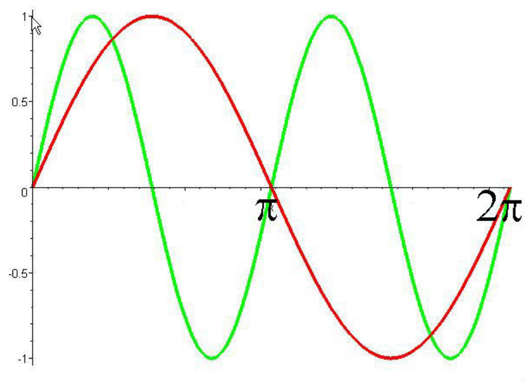


$$y = \sin x$$

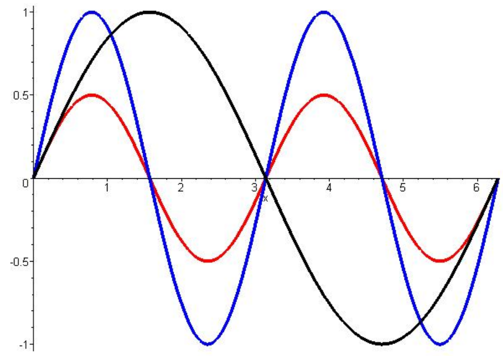


$$y = \sin x$$

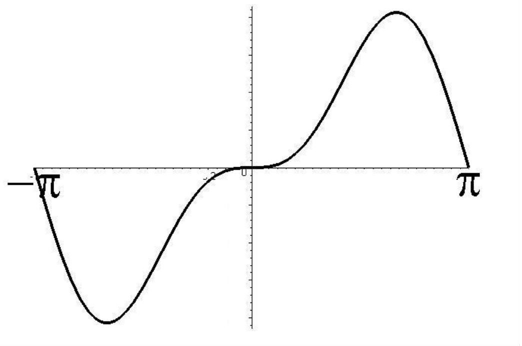
$$y = \sin 2x$$



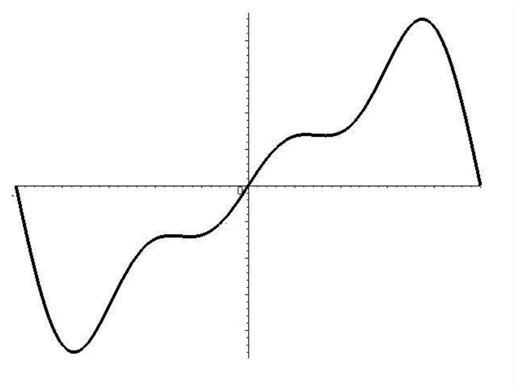
$$y = \sin x, \quad y = \sin 2x \quad \text{and} \quad y = \frac{1}{2}\sin 2x$$



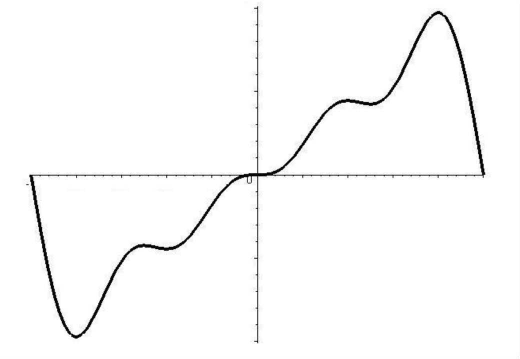
$$y = \sin x - \frac{1}{2}\sin 2x$$



$$y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$$

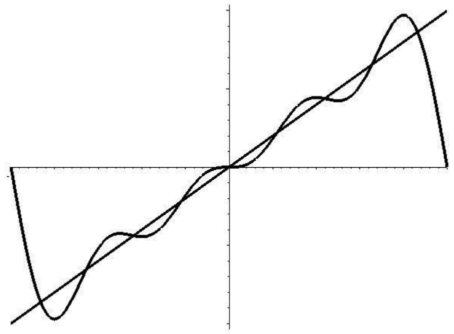


$$y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x$$

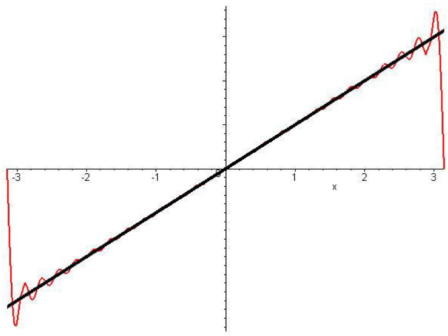


$$y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x$$

$$y = \frac{x}{2}$$



$$y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \cdots$$



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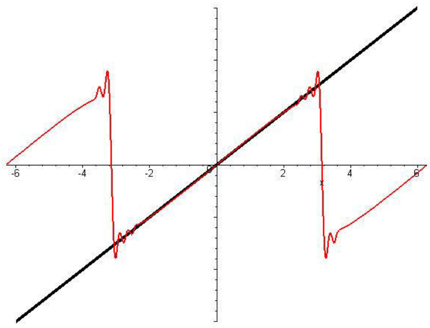

Fourier Sine Series

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots$$

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Fourier Cosine Series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx$$

More General Fourier Series

$$f(x) = \sum (a_n \cos nx + b_n \sin nx)$$

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$$f(x) = \sum (a_n \cos nx + b_n \sin nx)$$

The formulas for a_n and b_n are given by *integrals*

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

Find the coefficient b_1

$$f(x) = b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

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$$f(x) \sin x = b_1 \sin x \sin x + b_2 \sin x \sin 2x + b_3 \sin x \sin 3x + \cdots$$

Find the coefficient b_1

$$f(x) = b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$\int_{-\pi}^{\pi} f(x) \sin x \, dx = b_1 \int_{-\pi}^{\pi} \sin x \sin x \, dx + b_2 \int_{-\pi}^{\pi} \sin x \sin 2x \, dx + b_3 \int_{-\pi}^{\pi} \text{s}$$

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$$\int_{-\pi}^{\pi} f(x) \sin x \, dx = b_1 \cdot \pi + b_2 \cdot 0 + b_3 \cdot 0 + \cdots$$

Find the coefficient b_1

$$f(x) = b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$\int_{-\pi}^{\pi} f(x) \sin x \, dx = b_1 \int_{-\pi}^{\pi} \sin x \sin x \, dx + b_2 \int_{-\pi}^{\pi} \sin x \sin 2x \, dx + b_3 \int_{-\pi}^{\pi} \mathrm{s}$$

$$\int_{-\pi}^{\pi} f(x) \sin x \, dx = b_1 \cdot \pi + b_2 \cdot 0 + b_3 \cdot 0 + \cdots$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x \, dx$$

Find the coefficient b_2

$$f(x) = b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

Find the coefficient b_2

$$f(x) = b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$f(x) \sin 2x = b_1 \sin x \sin 2x + b_2 \sin 2x \sin 2x + b_3 \sin 2x \sin 3x + \cdots$$

Find the coefficient b_2

$$f(x) = b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$f(x) \sin 2x = b_1 \sin x \sin 2x + b_2 \sin 2x \sin 2x + b_3 \sin 2x \sin 3x + \cdots$$

$$\int_{-\pi}^{\pi} f(x) \sin 2x \, dx = b_1 \int_{-\pi}^{\pi} \sin 1x \sin 2x \, dx + b_2 \int_{-\pi}^{\pi} \sin 2x \sin 2x \, dx + b_3 \int_{-\pi}^{\pi} \text{si}$$

Find the coefficient b_2

$$f(x) = b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$f(x)\sin 2x = b_1 \sin x \sin 2x + b_2 \sin 2x \sin 2x + b_3 \sin 2x \sin 3x + \cdots$$

$$\int_{-\pi}^{\pi} f(x)\sin 2x\,dx = b_1 \underbrace{\int_{-\pi}^{\pi} \sin 1x \sin 2x\,dx}_{=0} + b_2 \underbrace{\int_{-\pi}^{\pi} \sin 2x \sin 2x\,dx}_{=\pi} + b_3 \underbrace{\int_{-\pi}}_{=0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$f(x) = \frac{x}{2}$$

Find the Fourier sine series

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Find the Fourier sine series

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{2} \sin nx \, dx$$

Use the integration by parts formula: $\int u \, dv = uv - \int v \, du$

$$u = \frac{x}{2} \qquad dv = \sin nx \, dx$$

$$du = \frac{1}{2} \, dx \qquad v = -\frac{1}{n} \cos nx$$

$$f(x) = \frac{x}{2}$$

Find the Fourier sine series

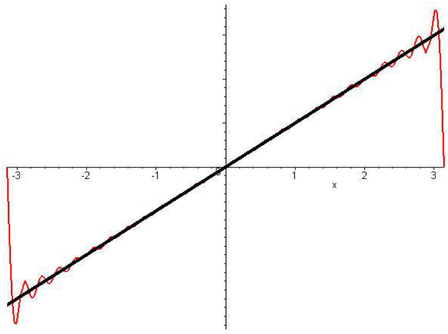
$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{2} \sin nx \, dx \\ &= \left[-\frac{x}{2\pi n} \cos nx \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{1}{2\pi n} \cos nx \, dx \\ &= -\frac{\pi}{n} \cos n\pi + 0 \end{aligned}$$

$$f(x) = \frac{x}{2}$$

Find the Fourier sine series

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2}\sin nx\,dx \\
&= \left[-\frac{x}{2\pi n}\cos nx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{1}{2\pi n}\cos nx\,dx \\
&= -\frac{1}{n}\cos n\pi + 0 \\
&= -\frac{1}{n}(-1)^n
\end{aligned}$$

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{-1}{n}(-1)^n \sin nx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$



What functions are represented by a Fourier sine series?

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
f(-x) &= \sum_{n=1}^{\infty} b_n \sin(-nx) \\
&= \sum_{n=1}^{\infty} b_n(-\sin nx) \\
&= -\sum_{n=1}^{\infty} b_n \sin nx \\
&= -f(x)
\end{aligned}$$

A function is called an *odd function* if:

$$f(-x) = -f(x)$$

Only odd functions are represented by Fourier sine series.

A function is called an *even function* if:

$$f(-x) = f(x)$$

Suppose $f(x)$ is given by a Fourier cosine series:

$$f(x) = \sum_n a_n \cos nx$$

$$
\begin{aligned}
f(x) &= \sum_n a_n \cos nx \\
f(-x) &= \sum_n a_n \cos(-nx) \\
&= \sum_n a_n \cos nx) \\
&= f(x)
\end{aligned}
$$

Conclusion: Only even functions are represented by Fourier cosine series