Even And Odd Functions

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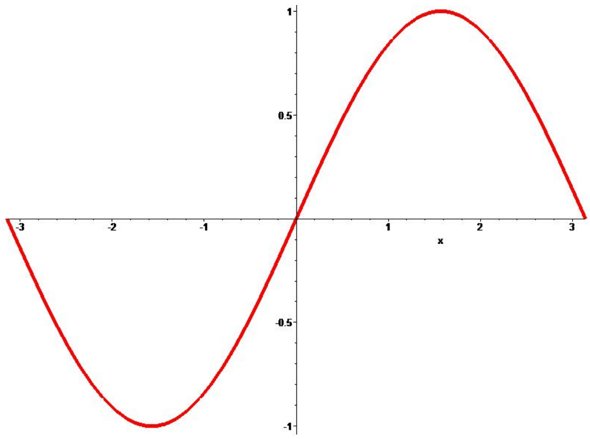
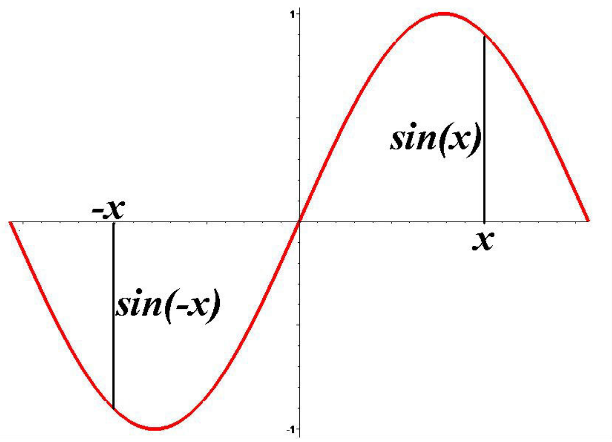


Given a function $f(x)$ find coefficients a_n and b_n so that:

$$f(x) = \sum_n (a_n \cos nx + b_n \sin nx)$$

$$\sin(-x) = -\sin(x)$$



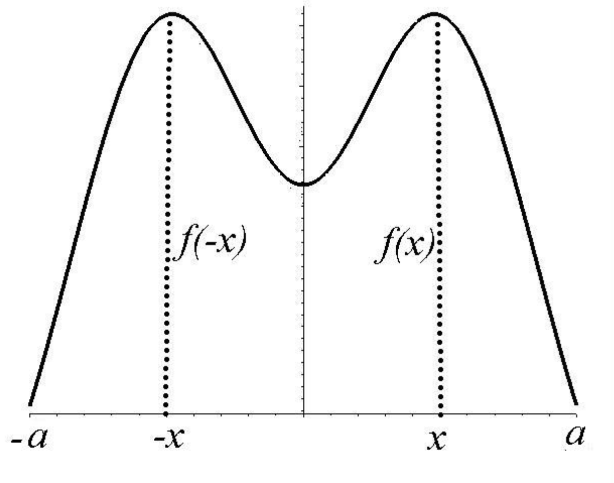
A function is called an *odd function* if

$$f(-x) = -f(x)$$

A function is called an *even function* if:

$$f(-x) = f(x)$$

Even function



Suppose $f_1(x)$ and $f_2(x)$ are both odd functions

$$f_1(-x) = -f_1(x) \qquad f_2(-x) = -f_2(x)$$

Let $g(x)$ be the product of these functions

$$g(x) = f_1(x) f_2(x)$$

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$$g(-x) = f_1(-x)f_2(-x) = (-f_1(x))(-f_2(x)) = f_1(x)f_2(x)$$

$$g(-x) = g(x)$$

Therefore, the product of two odd functions will result in an even function.
Example: $g(x) = \sin 2x \sin 3x$

Suppose $f_1(x)$ and $f_2(x)$ are both even functions

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$$g(x) = f_1(x)f_2(x)$$

$$g(-x) = f_1(-x)f_2(-x) = f_1(x)f_2(x) = g(x)$$

Therefore, the product of two even functions will result in an even function.
Example: $g(x) = \cos 2x \cos 3x$

Suppose $f_1(x)$ is odd and $f_2(x)$ is even

$$f_1(-x) = -f_1(x) \qquad f_2(-x) = f_2(x)$$

Let $g(x)$ be the product of these functions

$$g(x) = f_1(x)f_2(x)$$

Suppose $f_1(x)$ is odd and $f_2(x)$ is even

$$f_1(-x) = -f_1(x) \qquad f_2(-x) = f_2(x)$$

Let $g(x)$ be the product of these functions

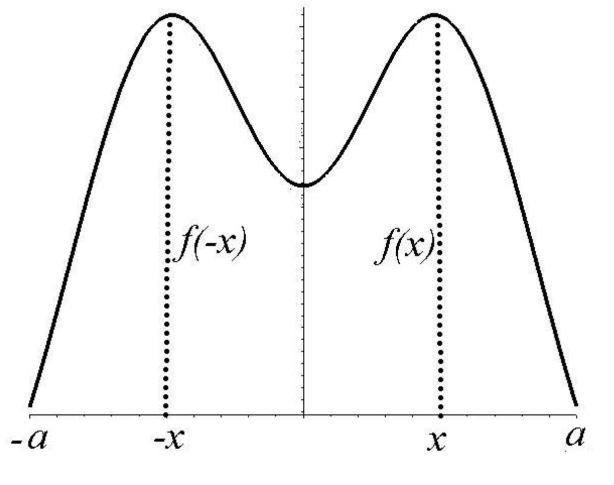
$$g(x) = f_1(x)f_2(x)$$

$$g(-x) = f_1(-x)f_2(-x) = -f_1(x)f_2(x) = -g(x)$$

Therefore, the product of odd and even functions will result in an odd function.
Example: $g(x) = \sin 2x \cos 3x$

Suppose $f(x)$ is an even function.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$



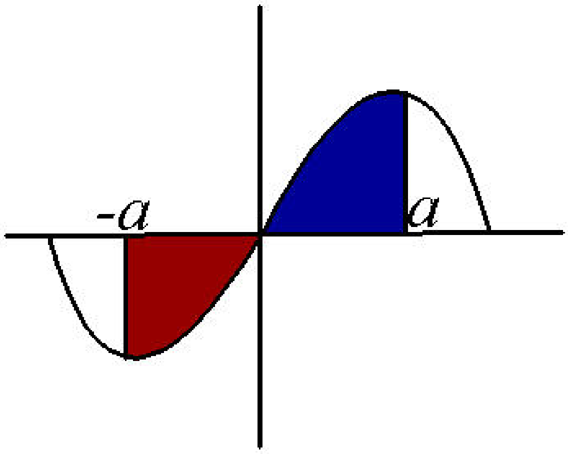
Suppose $f(x)$ is an even function.

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin nx \sin mx \, dx &= 2\int_{0}^{\pi} \sin nx \sin mx \, dx \\ &= \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}\end{aligned}$$

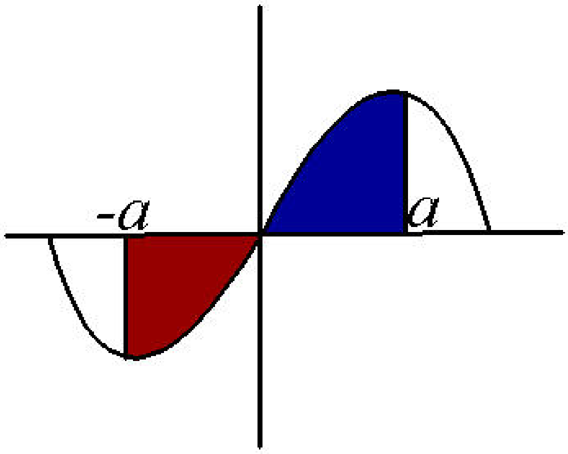
Suppose $f(x)$ is an odd function

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

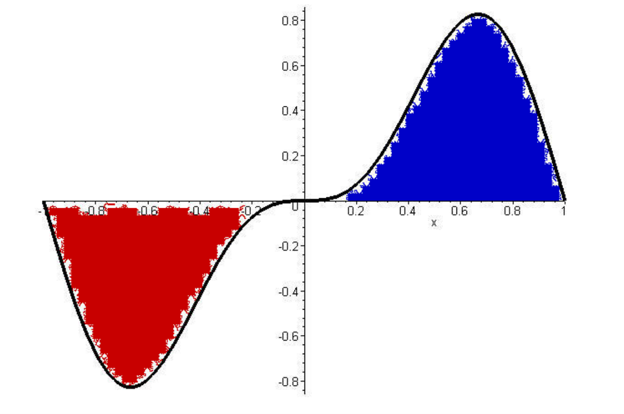


Suppose $f(x)$ is an odd function

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$



$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0$$



$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Let $G_n(x) = a_n \cos nx + b_n \sin nx$

$$
\begin{aligned}
G_n(x)\sin nx &= (a_n\cos nx + b_n\sin nx)\sin nx \\
&= a_n\cos nx\sin nx + b_n\sin nx\sin nx
\end{aligned}
$$

$\int_{-\pi}^{\pi} G_n(x)\sin nx\,dx$ will be the sum of two integrals:

$$
a_n\int_{-\pi}^{\pi}\cos nx\sin nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\sin nx\,dx
$$

$$
= a_n\cdot 0 + b_n\int_{-\pi}^{\pi}\sin nx\sin nx\,dx
$$

Let $G_n(x) = a_n \cos nx + b_n \sin nx$

$$\int_{-\pi}^{\pi} G_n(x) \sin nx \, dx = b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx$$

Similarly,

$$\int_{-\pi}^{\pi} G_n(x) \cos nx \, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx \, dx$$

Let $G_n(x) = a_n \cos nx + b_n \sin nx$

$$\int_{-\pi}^{\pi} G_n(x) \sin nx \, dx = b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx$$

Similarly,

$$\int_{-\pi}^{\pi} G_n(x) \cos nx \, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx \, dx$$

What if these are different from each other?

If $k \neq n$, then $\int_{-\pi}^{\pi} G_k(x) \sin nx \, dx = 0$

$$a_k \int_{-\pi}^{\pi} \cos kx \sin nx \, dx + b_k \int_{-\pi}^{\pi} \sin kx \sin nx \, dx = 0$$

Similarly, if $k \neq n$, then $\int_{-\pi}^{\pi} G_k(x) \cos nx \, dx = 0$

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Let $G_n(x) = a_n \cos nx + b_n \sin nx$

$$f(x) = G_0(x) + G_1(x) + \cdots + G_n(x) + \cdots$$

$$f(x) = G_0(x) + G_1(x) + \cdots + G_n(x) + \cdots = \sum_{k=0}^{\infty} G_k(x)$$

Multiply both sides by $\sin nx$

$$f(x) \sin nx = \sum_{k=0}^{\infty} G_k(x) \sin nx$$

Now integrate from $-\pi$ to π

$\int_{-\pi}^{\pi} f(x) \sin nx \, dx$ will be a sum of integrals:

$$\sum_{k=0}^{\infty} \int_{-\pi}^{\pi} G_k(x) \sin nx \, dx$$

$\int_{-\pi}^{\pi} f(x) \sin nx \, dx$ will be a sum of integrals:

$$\sum_{k \neq n} \int_{-\pi}^{\pi} G_k(x) \sin nx \, dx + \int_{-\pi}^{\pi} G_n(x) \sin nx \, dx$$

$\int_{-\pi}^{\pi} f(x) \sin nx \, dx$ will be a sum of integrals:

$$\sum_{k \neq n} \int_{-\pi}^{\pi} G_k(x) \sin nx \, dx + \int_{-\pi}^{\pi} G_n(x) \sin nx \, dx$$

$$= 0 + b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx$$

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx$$

If $n = 0$, this integral is just 0. Otherwise,

$$\begin{aligned} b_n &= \frac{1}{\int_{-\pi}^{\pi} \sin^2 nx \, dx} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{aligned}$$

Similarly,

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n \int_{-\pi}^{\pi} \cos^2 nx\,dx$$

$$a_n = \frac{1}{\int_{-\pi}^{\pi} \cos^2 nx\,dx} \int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

If $n \neq 0$ then,

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi$$

If $n = 0$ then

$$\int_{-\pi}^{\pi} \cos^2 0x \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$a_n = \frac{1}{\int_{-\pi}^{\pi} \cos^2 nx\, dx} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

If $n \neq 0$ then

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

If $n = 0$ then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos 0x\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

This way, every a_n is given by the formula:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$