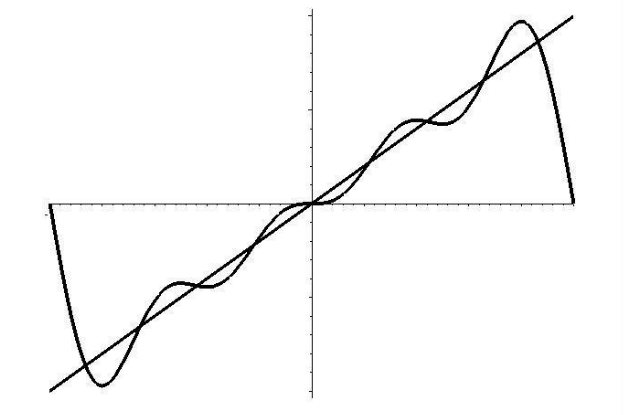
Fourier Series Examples

Dr. Elliott Jacobs

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$



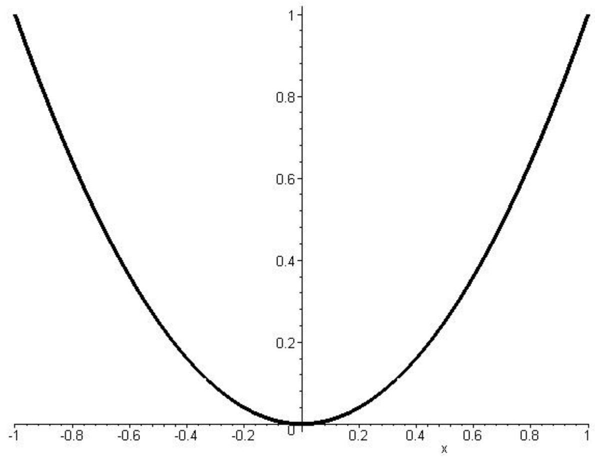
Formulas for Fourier Coefficients

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Find the Fourier series for $f(x) = x^2$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \, dx$$

For $n = 0$,

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos 0x \, dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \, dx = \frac{2}{3}\pi^2$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} a_n \cos nx$$

Now, calculate $a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$ for $n > 0$

$$\int u\,dv = uv - \int v\,du$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx \\ &= \frac{2\pi \sin \pi n}{n} - \frac{4}{\pi n}\int_0^{\pi} x \sin nx\,dx \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\,dx \\
&= \frac{2\pi \sin \pi n}{n} - \frac{4}{\pi n} \int_0^{\pi} x \sin nx\,dx \\
&= 0 - \frac{4}{\pi n} \int_0^{\pi} x \sin nx\,dx
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

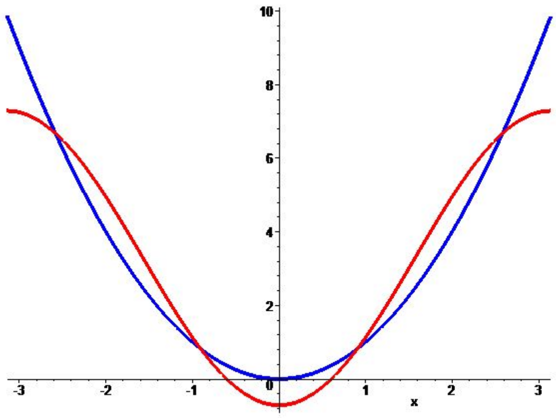
$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx \\
&= \frac{2\pi\sin\pi n}{n} - \frac{4}{\pi n}\int_0^\pi x\sin nx\,dx \\
&= 0 - \frac{4}{\pi n}\int_0^\pi x\sin nx\,dx \\
&= -\frac{4}{\pi n}\left(\frac{-\pi\cos\pi n}{n} + \int_0^\pi \frac{\cos nx}{n}\,dx\right)
\end{aligned}$$

$$
\begin{aligned}
a_n &= -\frac{4}{\pi n}\left(\frac{-\pi\cos\pi n}{n} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right) \\
&= -\frac{4}{\pi n}\left(\frac{-\pi(-1)^n}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right) \\
&= \frac{4}{n^2}(-1)^n
\end{aligned}
$$

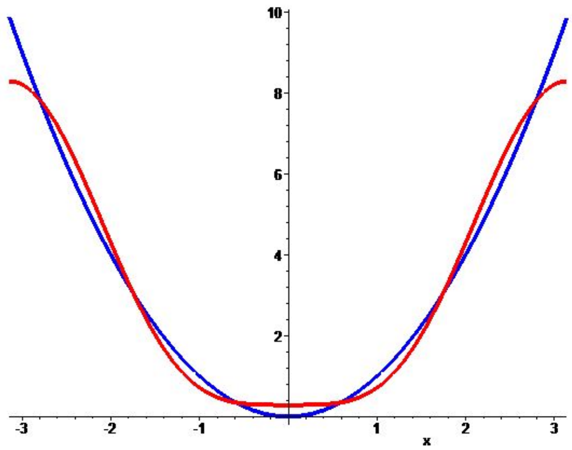
$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$
\begin{aligned}
x^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx \\
&= \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \cdots
\end{aligned}
$$

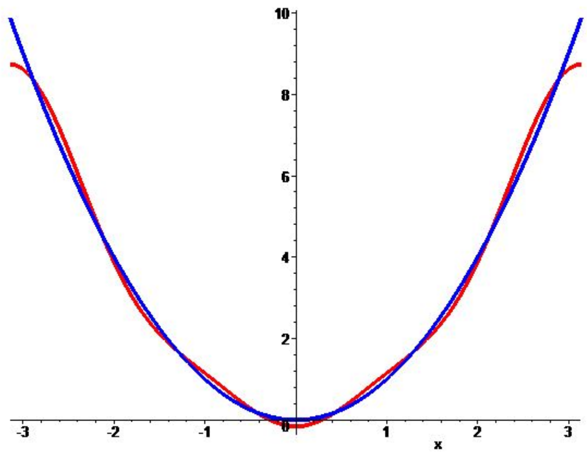
$$y = \frac{\pi^2}{3} - 4\cos x$$



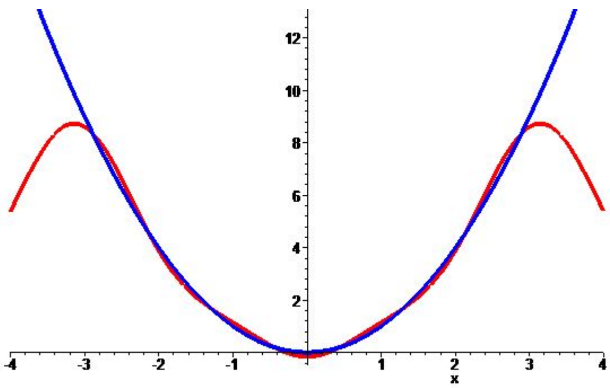
$$y = \frac{\pi^2}{3} - 4\cos x + \cos 2x$$



$$y = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x$$



$$y = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x$$



Calculus II: Test convergence of the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Calculus II: Test convergence of the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{T\to\infty} \int_1^{T} \frac{1}{x^2}\,dx = \lim_{T\to\infty} \left(1 - \frac{1}{T}\right) = 1$$

If $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges, then by the Integral Test, $\sum \frac{1}{n^2}$ converges also.

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

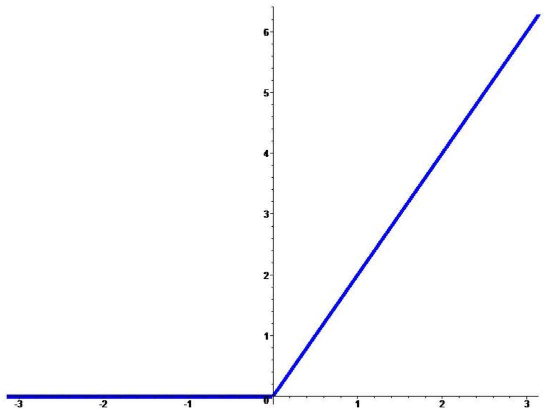
Substitute $x = \pi$. Remember that $\cos n\pi = (-1)^n$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n(-1)^n$$

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

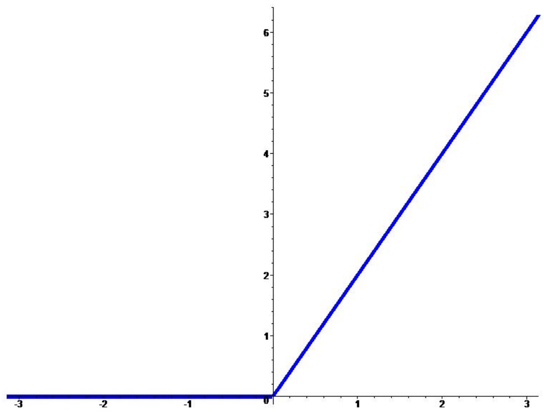
$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Find the Fourier series for $f(x) = x + |x|$



$f(x) = 0$ for $x < 0$ and $f(x) = 2x$ for $x \geq 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+|x|)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} 2x\sin nx\,dx$$



$$
\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^{\pi} 2x \sin nx \, dx \\
&= \frac{1}{\pi} \left(\left[-\frac{2x}{n} \cos nx \right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} \, dx \right)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} 2x\sin nx\,dx \\
&= \frac{1}{\pi}\left(\left[-\frac{2x}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi}\frac{-\cos nx}{n}\,dx\right) \\
&= \frac{1}{\pi}\left(-\frac{2\pi}{n}\cos n\pi + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right) \\
&= -\frac{2}{n}(-1)^n
\end{aligned}
$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + |x|) \cos nx \, dx = \frac{1}{\pi} \int_{0}^{\pi} 2x \cos nx \, dx$$

$$a_0 = \frac{1}{\pi} \int_{0}^{\pi} 2x \, dx = \pi$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^\pi 2x \cos nx \, dx \\ &= \frac{1}{\pi} \left(\left[\frac{2x \sin nx}{n} \right]_0^\pi - \int_0^\pi \frac{2 \sin nx}{n} \, dx \right) \end{aligned}$$

For $n > 0$,

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{\pi} 2x \cos nx \, dx \\
&= \frac{1}{\pi} \left(\left[\frac{2x \sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{2 \sin nx}{n} \, dx \right) \\
&= \frac{1}{\pi} \left(0 + \left[\frac{2 \cos nx}{n^2} \right]_0^{\pi} \right) \\
&= \frac{2}{\pi n^2} \left((-1)^n - 1 \right)
\end{aligned}
$$

The Fourier series for $x + |x|$ is:

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n^2} \left((-1)^n - 1 \right) \cos nx - \frac{2}{n} (-1)^n \sin nx \right)$$