Fourier Series on the interval $-L \leq x \leq L$

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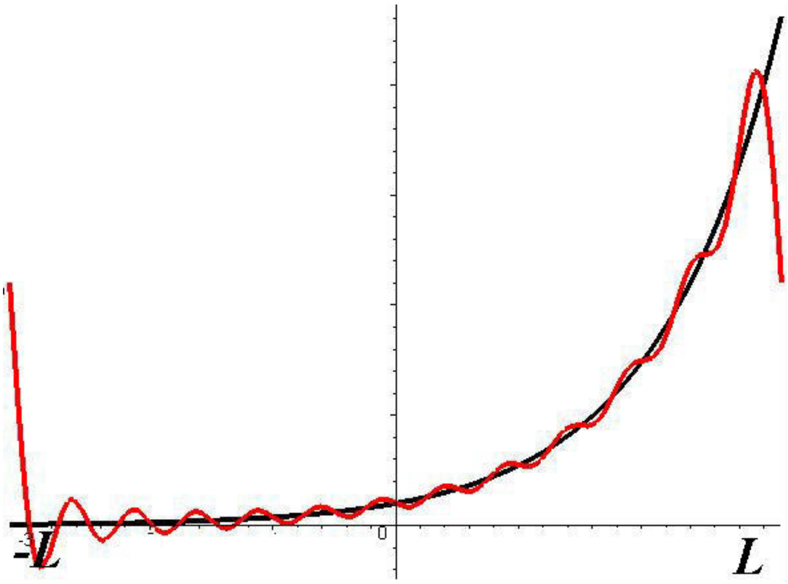


$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\, dx$$

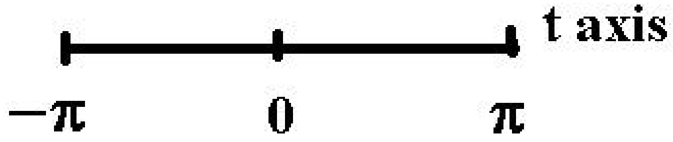
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\, dx$$

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

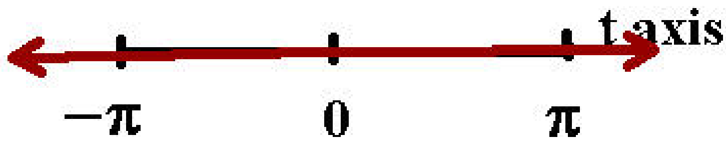
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) \cos nt \, dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) \sin nt \, dt$$

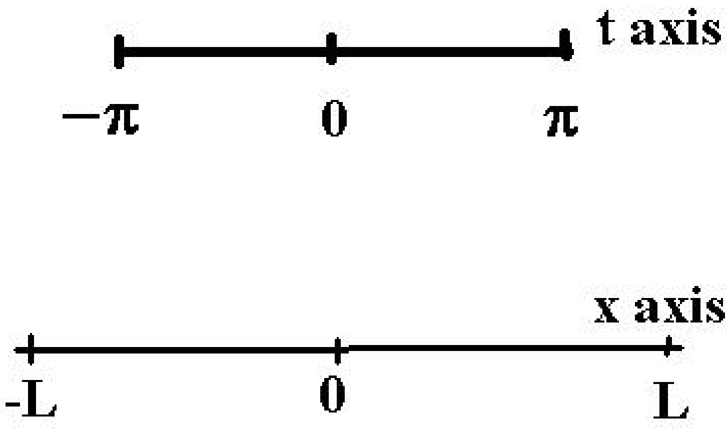
$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$ represents our function for $-\pi \leq t \leq \pi$



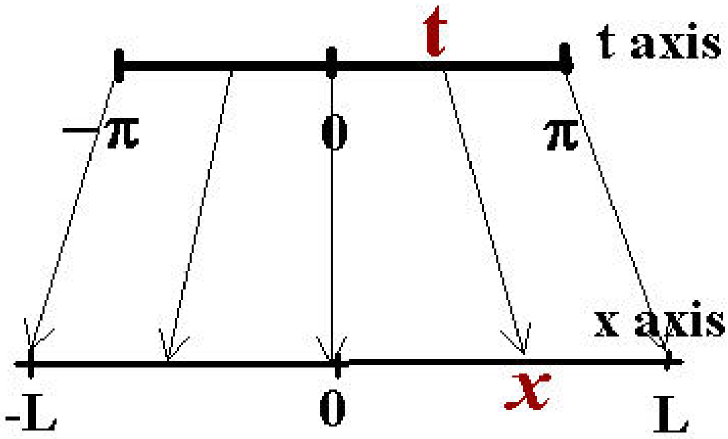
Let's expand our interval



We want our Fourier series to in terms of x where $-L \leq x \leq L$



$$x = \frac{L}{\pi} t$$



If $x = \frac{L}{\pi} t$ then $t = \frac{\pi}{L} x$

$$g(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$g\left(\frac{\pi x}{L}\right) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$
Let $f(x) = g\left(\frac{\pi x}{L}\right) = g(t)$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\,dt$$

$$t = \frac{\pi x}{L} \qquad dt = \frac{\pi}{L}\,dx$$

The limits change from $\int_{-\pi}^{\pi}(\quad)\,dt$ to $\int_{-L}^{L}(\quad)\,dx$

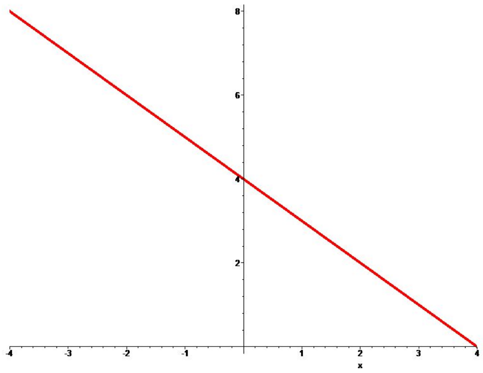
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos\frac{n\pi x}{L}\cdot\frac{\pi}{L}\,dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \end{aligned}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

Find the Fourier series for $f(x) = 4 - x$ on the interval $-4 \leq x \leq 4$



$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{4} + b_n \sin\frac{n\pi x}{4} \right)$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(x)\cos\frac{n\pi x}{4}\,dx$$

$$b_n = \frac{1}{4}\int_{-4}^{4} f(x)\sin\frac{n\pi x}{4}\,dx$$

$$a_n = \frac{1}{4}\int_{-4}^{4}(4-x)\cos\frac{n\pi x}{4}\,dx$$

If $n = 0$ then:

$$a_0 = \frac{1}{4}\int_{-4}^{4}(4-x)\,dx = \frac{1}{4}\left[4x - \frac{1}{2}x^2\right]_{-4}^{4} = 8$$

If $n > 0$ then;

$$
\begin{aligned}
a_n &= \frac{1}{4} \int_{-4}^{4} (4-x) \cos \frac{n\pi x}{4} \, dx \\
&= \frac{1}{4} \left(\left[\frac{(4-x)4}{n\pi} \sin \frac{n\pi x}{4} \right]_{-4}^{4} - \int_{-4}^{4} (-1) \frac{4}{n\pi} \sin \frac{n\pi x}{4} \right) dx
\end{aligned}
$$

If $n > 0$ then;

$$
\begin{aligned}
a_n &= \frac{1}{4} \int_{-4}^{4} (4 - x) \cos \frac{n\pi x}{4} \, dx \\
&= \frac{1}{4} \left(\left[\frac{(4-x)4}{n\pi} \sin \frac{n\pi x}{4} \right]_{-4}^{4} - \int_{-4}^{4} (-1) \frac{4}{n\pi} \sin \frac{n\pi x}{4} \right) \, dx \\
&= 0
\end{aligned}
$$

$$f(x) = 4 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{4}$$

$$b_n = \frac{1}{4} \int_{-4}^{4} (4 - x) \sin \frac{n\pi x}{4} \, dx$$

$$f(x) = 4 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{4}$$

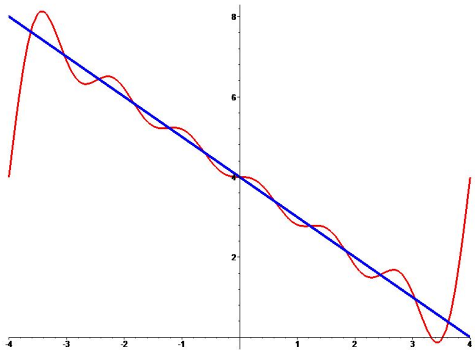
$$\begin{aligned} b_n &= \frac{1}{4}\int_{-4}^{4} (4 - x)\sin\frac{n\pi x}{4}\,dx \\ &= \frac{1}{4}\int_{-4}^{4} 4\sin\frac{n\pi x}{4}\,dx - \frac{1}{4}\int_{-4}^{4} x\sin\frac{n\pi x}{4}\,dx \end{aligned}$$

$$f(x) = 4 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{4}$$

$$\begin{aligned}
b_n &= \frac{1}{4} \int_{-4}^{4} (4 - x) \sin \frac{n\pi x}{4} \, dx \\
&= \frac{1}{4} \int_{-4}^{4} 4 \sin \frac{n\pi x}{4} \, dx - \frac{1}{4} \int_{-4}^{4} x \sin \frac{n\pi x}{4} \, dx \\
&= 0 - \frac{1}{4} \cdot 2 \int_{0}^{4} x \sin \frac{n\pi x}{4} \, dx
\end{aligned}$$

$$
\begin{aligned}
b_n &= -\frac{1}{2}\int_0^4 x\sin\frac{n\pi x}{4}\,dx \\
&= -\frac{1}{2}\left(\left[\frac{-4x}{n\pi}\cos\frac{n\pi x}{4}\right]_0^4 - \int_0^4 -\frac{4}{n\pi}\cos\frac{n\pi x}{4}\,dx\right) \\
&= \frac{8}{n\pi}(-1)^n
\end{aligned}
$$

$$f(x) = 4 + \sum_{n=1}^{\infty} \frac{8}{n\pi}(-1)^n \sin\frac{n\pi x}{4}$$



$$\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

$\int_{-L}^{L} f(x)g(x) \, dx$ will be called an *inner product*

$$(f,\ g) = \int_{-L}^{L} f(x)g(x) \, dx$$

$$\vec{\mathbf{u}} = \langle u_1,\ u_2 \rangle \qquad \vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle$$

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2$$

If we take $\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}$, we get a non-negative number

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{u}} = u_1^2 + u_2^2 \geq 0$$

$$(f,\ g) = \int_{-L}^{L} f(x)g(x)\,dx$$

If we were to take the inner product of f with itself, we get a non-negative number

$$(f,\ f) = \int_{-L}^{L} (f(x))^2\,dx \geq 0$$

The dot product is commutative

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \vec{\mathbf{v}} \bullet \vec{\mathbf{u}}$$

The inner product is commutative

$$
\begin{aligned}
(f,\ g) &= \int_{-L}^{L} f(x)g(x)\,dx \\
&= \int_{-L}^{L} g(x)f(x)\,dx \\
&= (g,\ f)
\end{aligned}
$$

$$\vec{\mathbf{u}} = \langle u_1,\ u_2 \rangle \qquad \vec{\mathbf{v}} = \langle v_1,\ v_2 \rangle \qquad \vec{\mathbf{w}} = \langle w_1,\ w_2 \rangle$$

Dot product obeys a distributive law:

$$(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \bullet \vec{\mathbf{w}} = \vec{\mathbf{u}} \bullet \vec{\mathbf{w}} + \vec{\mathbf{v}} \bullet \vec{\mathbf{w}}$$

Inner product obeys a distributive law

$$\begin{aligned}(\phi_1 + \phi_2,\ f) &= \int_{-L}^{L} (\phi_1(x) + \phi_2(x)) f(x)\, dx \\ &= \int_{-L}^{L} \phi_1(x) f(x)\, dx + \int_{-L}^{L} \phi_2(x) f(x)\, dx \\ &= (\phi_1,\ f) + (\phi_2,\ f)\end{aligned}$$

$$(\phi_1 + \phi_2 + \phi_3,\ f) = (\phi_1,\ f) + (\phi_2,\ f) + (\phi_3,\ f)$$

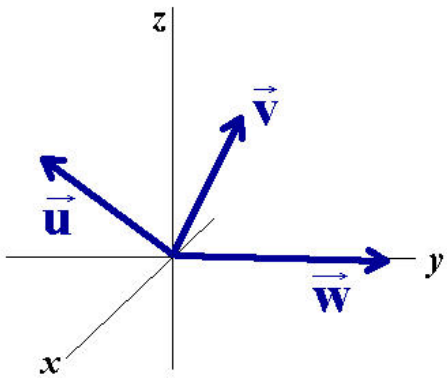
$$\left(\sum_k \phi_k,\ f\right) = \sum_k (\phi_k,\ f)$$

A set of vectors $\{\vec{\mathbf{u}},\ \vec{\mathbf{v}},\ \vec{\mathbf{w}}\}$ is an *orthogonal set* if:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = 0 \qquad \vec{\mathbf{u}} \bullet \vec{\mathbf{w}} = 0 \qquad \vec{\mathbf{v}} \bullet \vec{\mathbf{w}} = 0$$

Example:

$$\vec{\mathbf{u}} = \langle 1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{w}} = \langle 0,\ 2,\ 0 \rangle$$



Two functions f and g are orthogonal on $[-L,\ L]$ if $(f,\ g) = 0$

$$\int_{-L}^{L} f(x)g(x)\,dx = 0$$

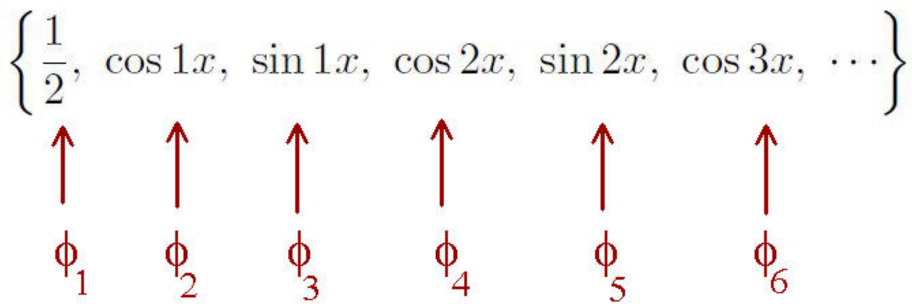
If $n \neq m$

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = 0$$

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = 0$$

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = 0$$

An orthogonal set:



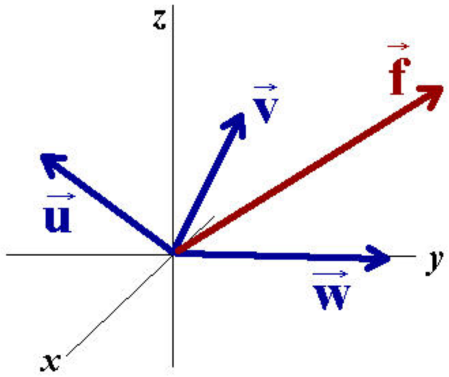
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

This can be written as:

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

$$\vec{\mathbf{u}} = \langle 1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{w}} = \langle 0,\ 2,\ 0 \rangle$$

$$\vec{\mathbf{f}} = \langle 0,\ 8,\ 4 \rangle$$



$$\vec{\mathbf{u}} = \langle 1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{w}} = \langle 0,\ 2,\ 0 \rangle$$

$$\vec{\mathbf{f}} = \langle 0,\ 8,\ 4 \rangle$$

$$\vec{\mathbf{f}} = c_1 \vec{\mathbf{u}} + c_2 \vec{\mathbf{v}} + c_3 \vec{\mathbf{w}}$$

$$\vec{\mathbf{u}} = \langle 1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{w}} = \langle 0,\ 2,\ 0 \rangle$$

$$\vec{\mathbf{f}} = \langle 0,\ 8,\ 4 \rangle$$

$$\vec{\mathbf{f}} = c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}$$

$$\begin{aligned} \vec{\mathbf{f}} \bullet \vec{\mathbf{u}} &= (c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}) \bullet \vec{\mathbf{u}} \\ &= c_1\vec{\mathbf{u}} \bullet \vec{\mathbf{u}} + c_2\vec{\mathbf{v}} \bullet \vec{\mathbf{u}} + c_3\vec{\mathbf{w}} \bullet \vec{\mathbf{u}} \\ &= c_1\vec{\mathbf{u}} \bullet \vec{\mathbf{u}} + c_2 \cdot 0 + c_3 \cdot 0 \end{aligned}$$

$$\text{Therefore:} \qquad c_1 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}}$$

$$\vec{\mathbf{u}} = \langle 1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{w}} = \langle 0,\ 2,\ 0 \rangle$$

$$\vec{\mathbf{f}} = \langle 0,\ 8,\ 4 \rangle$$

$$\vec{\mathbf{f}} = c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}$$

$$\begin{aligned} \vec{\mathbf{f}} \bullet \vec{\mathbf{v}} &= (c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}) \bullet \vec{\mathbf{v}} \\ &= c_1\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} + c_2\vec{\mathbf{v}} \bullet \vec{\mathbf{v}} + c_3\vec{\mathbf{w}} \bullet \vec{\mathbf{v}} \\ &= c_1 \cdot 0 + c_2\vec{\mathbf{v}} \bullet \vec{\mathbf{v}} + c_3 \cdot 0 \end{aligned}$$

$$\text{Therefore:} \qquad c_2 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{v}}}{\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}}$$

$$\vec{\mathbf{u}} = \langle 1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{w}} = \langle 0,\ 2,\ 0 \rangle$$

$$\vec{\mathbf{f}} = \langle 0,\ 8,\ 4 \rangle$$

$$\vec{\mathbf{f}} = c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}$$

$$\begin{aligned} \vec{\mathbf{f}} \bullet \vec{\mathbf{w}} &= (c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}) \bullet \vec{\mathbf{w}} \\ &= c_1\vec{\mathbf{u}} \bullet \vec{\mathbf{w}} + c_2\vec{\mathbf{v}} \bullet \vec{\mathbf{w}} + c_3\vec{\mathbf{w}} \bullet \vec{\mathbf{w}} \\ &= c_1 \cdot 0 + c_2 \cdot 0 + c_3\vec{\mathbf{w}} \bullet \vec{\mathbf{w}} \end{aligned}$$

$$\text{Therefore:} \qquad c_3 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{w}}}{\vec{\mathbf{w}} \bullet \vec{\mathbf{w}}}$$

$$\vec{\mathbf{u}} = \langle 1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{v}} = \langle -1,\ 0,\ 1 \rangle \qquad \vec{\mathbf{w}} = \langle 0,\ 2,\ 0 \rangle$$

$$\vec{\mathbf{f}} = \langle 0,\ 8,\ 4 \rangle$$

$$\vec{\mathbf{f}} = c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}$$

$$c_1 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}} = 2 \quad c_2 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{v}}}{\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}} = 2 \quad c_3 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{w}}}{\vec{\mathbf{w}} \bullet \vec{\mathbf{w}}} = 4$$

$$\vec{\mathbf{f}} = 2\vec{\mathbf{u}} + 2\vec{\mathbf{v}} + 4\vec{\mathbf{w}}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

This can be written as:

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

Find the coefficients

$$f = \sum_k c_k \phi_k$$

$$(f,\ \phi_n) = \left(\sum_k c_k \phi_k,\ \phi_n \right) = \sum_k (c_k \phi_k,\ \phi_n)$$

$$f = \sum_k c_k \phi_k$$

$$\begin{aligned}
(f,\ \phi_n) &= \left(\sum_k c_k \phi_k,\ \phi_n \right) = \sum_k (c_k \phi_k,\ \phi_n) \\
&= c_1(\phi_1,\ \phi_n) + c_2(\phi_2,\ \phi_n) + \cdots + c_n(\phi_n,\ \phi_n) + \cdots \\
&= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n(\phi_n,\ \phi_n) + \cdots \\
&= c_n(\phi_n,\ \phi_n)
\end{aligned}$$

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_k(x)$$

$$c_n = \frac{(f,\ \phi_n)}{(\phi_n,\ \phi_n)}$$

Compare the following sum of orthogonal vectors:

$$\vec{\mathbf{f}} = c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}}$$

$$c_1 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{u}}}{\vec{\mathbf{u}} \bullet \vec{\mathbf{u}}} \quad c_2 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{v}}}{\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}} \quad c_3 = \frac{\vec{\mathbf{f}} \bullet \vec{\mathbf{w}}}{\vec{\mathbf{w}} \bullet \vec{\mathbf{w}}}$$

with the following orthogonal expansion of functions:

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_k(x)$$

$$c_n = \frac{(f,\ \phi_n)}{(\phi_n,\ \phi_n)}$$