Fourier Series Solution of the Heat Equation

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Temperature: $u = u(x,t)$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions

$$u(0,t) = 0 \qquad\qquad u(L,t) = 0$$

Initial condition:

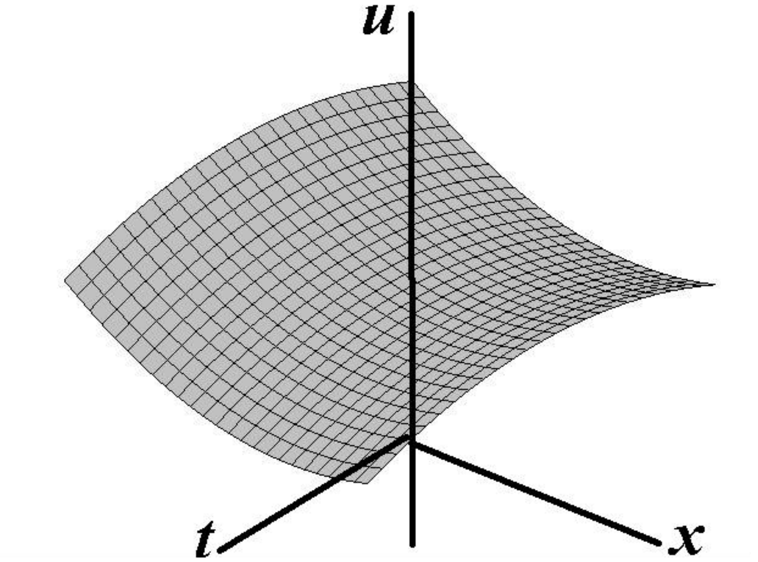
$$u(x,0) = f(x)$$

Trivial solution:

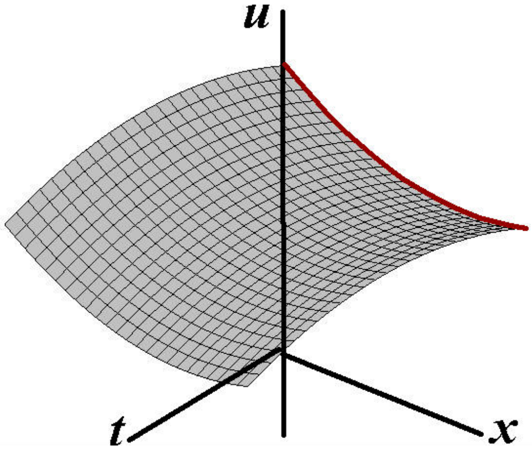
$$u(x,t) = 0 \quad \text{for all } x \text{ and } t$$

$$\frac{\partial}{\partial t}(0) = 0 = \alpha^2 \frac{\partial^2}{\partial x^2}(0)$$

$$u = u(x, t)$$

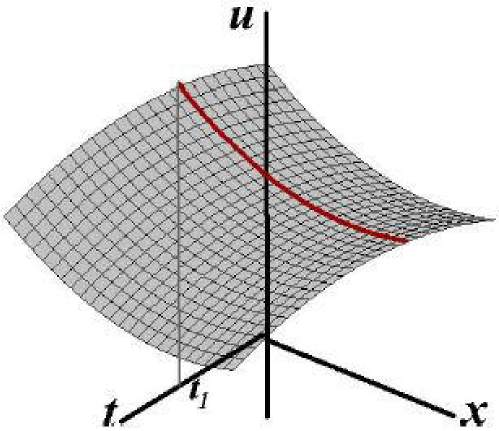


$$u(x,0) = \sum_n \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$



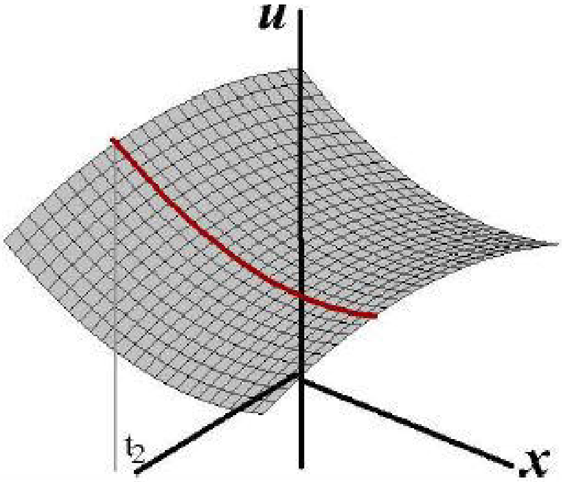
Need new coefficients at t_1

$$u(x,t_1) = \sum_n \left(\boxed{a_n}\cos\frac{n\pi x}{L} + \boxed{b_n}\sin\frac{n\pi x}{L}\right)$$



Coefficients change again at t_2

$$u(x,t_2) = \sum_n \left(\boxed{a_n} \cos \frac{n\pi x}{L} + \boxed{b_n} \sin \frac{n\pi x}{L} \right)$$



$$u(x,t) = \sum_n \left(\boxed{a_n(t) \cos \frac{n\pi x}{L}} + \boxed{b_n(t) \sin \frac{n\pi x}{L}} \right)$$

(function of t)(function of x)

Find all functions $X(x)$ and $T(t)$ so that

$$u = X(x)T(t)$$

solves

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

as well as the boundary conditions

$$u(x,t) = X(x)T(t)$$

The conditions $u(0,t) = 0$ and $u(L,t) = 0$ imply that

$$X(0)T(t) = 0 \qquad X(L)T(t) = 0$$

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If we want *nontrivial solutions* then $X(0) = 0$ and $X(L) = 0$

Substitute $u = X(x)T(t)$ into $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial}{\partial t}(X(x)T(t)) = \alpha^2 \frac{\partial^2}{\partial x^2}(X(x)T(t))$$

Substitute $u = X(x)T(t)$ into $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial}{\partial t}(X(x)T(t)) = \alpha^2 \frac{\partial^2}{\partial x^2}(X(x)T(t))$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Substitute $u = X(x)T(t)$ into $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial}{\partial t}(X(x)T(t)) = \alpha^2 \frac{\partial^2}{\partial x^2}(X(x)T(t))$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

$$\begin{array}{r} \textbf{depends on t} \rightarrow \\[2em] \textbf{depends on t} \rightarrow \end{array} \frac{T'(t)}{\alpha^2 T(t)} = \overset{\textbf{no dependence on t}}{\frac{X''(x)}{X(x)}}$$

no dependence on x

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} \begin{array}{l} \leftarrow \textbf{depends on x} \\[2em] \leftarrow \textbf{depends on x} \end{array}$$

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

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$$\frac{T'(t)}{\alpha^2 T(t)} = \lambda$$

$$\frac{1}{\alpha^2}\frac{d}{dt}(\ln T(t)) = \lambda$$

$$\ln T(t) = \int \alpha^2 \lambda \, dt = \alpha^2 \lambda t + C$$

$$T(t) = (\mathrm{Const}) e^{\alpha^2 \lambda t}$$

$$\frac{X''(x)}{X(x)} = \lambda$$

$$X''(x) - \lambda X(x) = 0$$

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$$X''(x) - \lambda X(x) = 0$$

If $\lambda = 0$ then:

$$X''(x) = 0$$

$$X(x) = ax + b$$

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$X(0) = 0$ implies that $b = 0$ so $X(x) = ax$
$X(L) = 0$ implies that $0 = aL$ so $a = 0$

$$X(x) = ax + b = 0x + 0$$

$$X(x) = 0 \quad \text{for all } x$$

$X(x)T(t) = 0 \cdot T(t) = 0$
This is the trivial solution

$$X''(x) - \lambda X(x) = 0$$

If we want nontrivial solutions then $\lambda \neq 0$
We can get solutions to this equation of the form e^{rx}

Substitute $X = e^{rx}$ into $X'' - \lambda X = 0$

$$\frac{d^2}{dx^2}\left(e^{rx}\right) - \lambda e^{rx} = 0$$

$$r^2 e^{rx} - \lambda e^{rx} = 0$$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

Both $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$ are solutions of $X'' - \lambda X = 0$. The general solution is:

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

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$$0 = X(0) = c_1 e^0 + c_2 e^0$$

$$c_2 = -c_1$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} - c_1 e^{-\sqrt{\lambda}x}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} - c_1 e^{-\sqrt{\lambda}x}$$

$$0 = X(L) = c_1 e^{\sqrt{\lambda}L} - c_1 e^{-\sqrt{\lambda}L}$$

$$0 = e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}$$

$$e^{\sqrt{\lambda}L} = e^{-\sqrt{\lambda}L}$$

$$e^{2\sqrt{\lambda}L} = 1$$

$$e^{2\sqrt{\lambda}L} = 1$$

This can only happen if the exponent is imaginary: $2\sqrt{\lambda}L = i\theta$

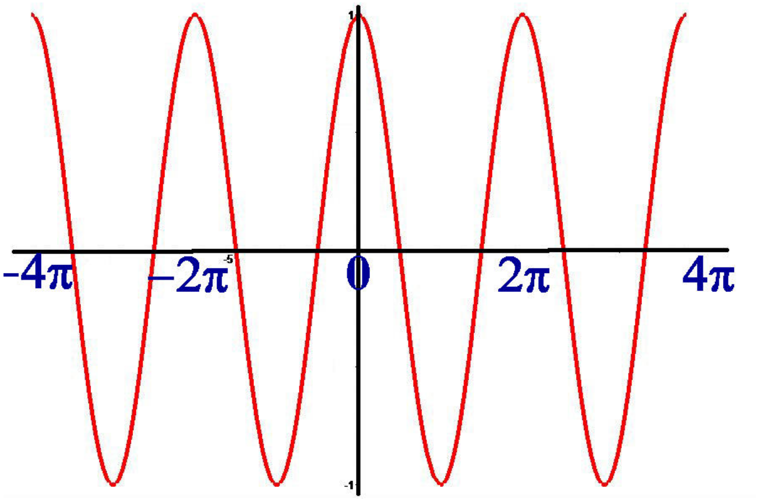
$$e^{i\theta} = 1$$

$$\cos\theta + i\sin\theta = 1$$

$$\cos\theta = 1 \qquad \sin\theta = 0$$

$\cos\theta = 1$ only at $\theta = 2n\pi$.

At these points, $\sin\theta = 0$



$$2\sqrt{\lambda}L = i\theta = 2\pi n i$$

$$\sqrt{\lambda}L = \pi n i$$

$$\sqrt{\lambda} = \frac{n\pi i}{L}$$

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{\lambda}x} - c_1 e^{-\sqrt{\lambda}x} \\ &= c_1 \left(e^{\frac{n\pi x}{L} i} - e^{-\frac{n\pi x}{L} i} \right) \end{aligned}$$

$$X(x) = c_1 \left(e^{\frac{n\pi x}{L} i} - e^{-\frac{n\pi x}{L} i}\right)$$

$$e^{\frac{n\pi x}{L} i} = \cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}$$

$$e^{-\frac{n\pi x}{L} i} = \cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}$$

$$X(x) = (\text{Const})\sin\frac{n\pi x}{L}$$

$$\sqrt{\lambda} = \frac{n\pi i}{L}$$

$$\lambda = -\frac{n^2\pi^2}{L^2}$$

$$T(t) = (\mathrm{Const})e^{\alpha^2\lambda t} = (\mathrm{Const})e^{-\frac{n^2\alpha^2\pi^2}{L^2}t}$$

$$X(x) = (\mathrm{Const})\sin\frac{n\pi x}{L}$$

$$X(x)T(t) = (\text{Const})e^{-\frac{n^2\alpha^2\pi^2}{L^2}t}\sin\frac{n\pi x}{L}$$

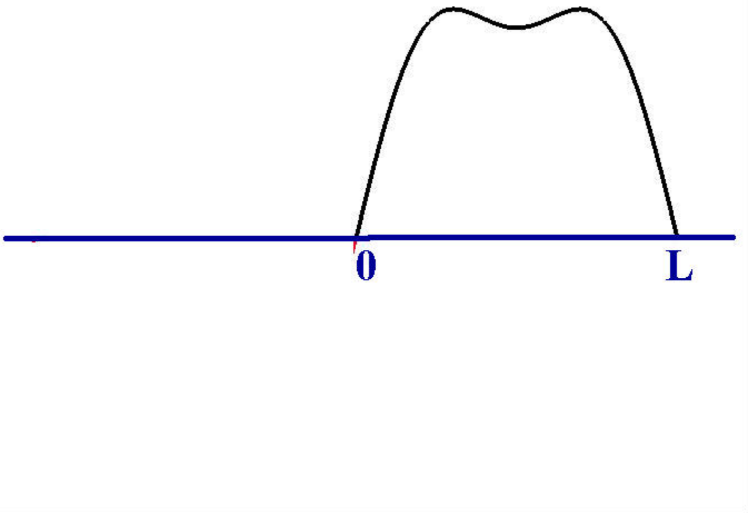
$$n = 1,\ 2,\ 3,\ \ldots$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\alpha^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L}$$

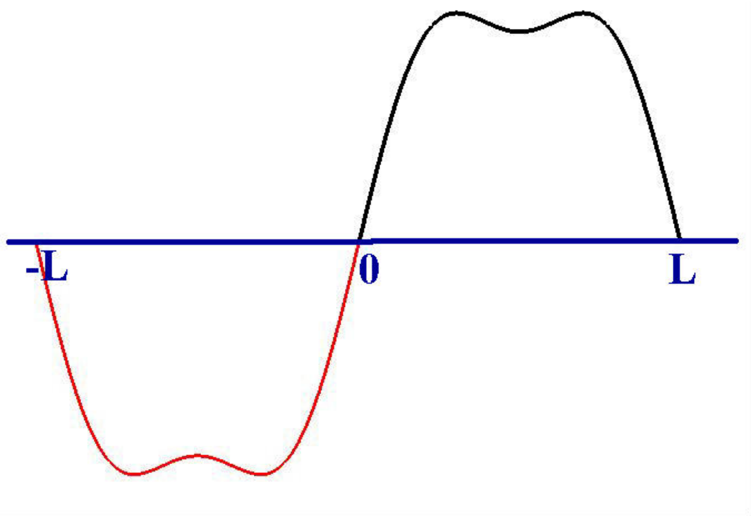
Finally, impose the initial condition $u(x,0) = f(x)$ for $0 \leq x \leq L$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$u(x,0) = f(x)$ is only specified from 0 to L



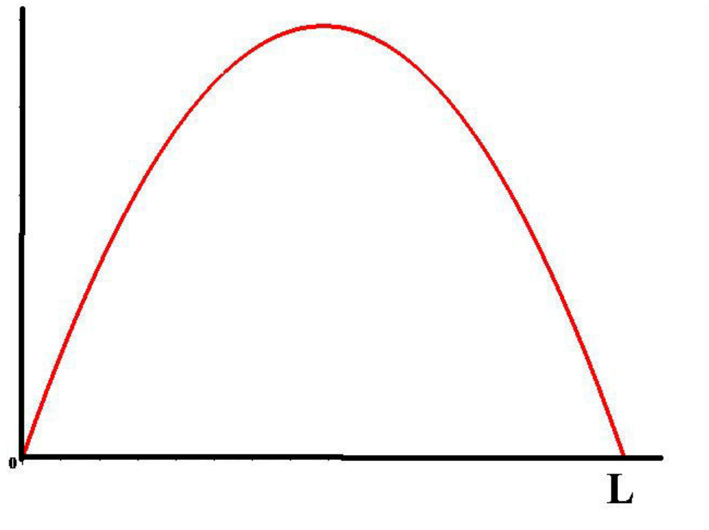
The Fourier sine series extends $f(x)$ to negative values of x in such a way that it is an odd function.



$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \\ &= \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \end{aligned}$$

$$f(x) = x(L - x)$$



$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L x(L-x)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{4L^2}{n^3\pi^3}\left(1-(-1)^n\right)
\end{aligned}
$$

$$
u(x,t) = \sum_{\text{odd } n} \frac{8L^2}{n^3\pi^3} e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}
$$

$$u = u(x, t)$$