Solution of the Heat Equation with Neumann boundary conditions

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$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

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Last time, we used the boundary conditions:

$$u(0,t) = 0 \qquad u(L,t) = 0$$

and initial condition $u(x,0) = x(L-x)$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

This time, take $L = 1$ and use the following the boundary conditions:

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(1, t) = 0$$

New initial condition: $u(x, 0) = \sin^2 \pi x$

Substitute $u = X(x)T(t)$ into $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial}{\partial t}(X(x)T(t)) = \alpha^2 \frac{\partial^2}{\partial x^2}(X(x)T(t))$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

Substitute $u = X(x)T(t)$ into $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial}{\partial t}(X(x)T(t)) = \alpha^2 \frac{\partial^2}{\partial x^2}(X(x)T(t))$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Substitute $u = X(x)T(t)$ into $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

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$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda^2$$

$$\frac{1}{\alpha^2}\frac{d}{dt}(\ln T(t)) = -\lambda^2$$

$$\ln T(t) = -\int \alpha^2\lambda^2 \, dt = -\alpha^2\lambda^2 t + C$$

$$T(t) = (\text{Const})e^{-\alpha^2\lambda^2 t}$$

$$\frac{X''(x)}{X(x)} = -\lambda^2$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$\text{If } \lambda = 0 \text{ then:} \quad X''(x) = 0$$

$$X(x) = c_1 + c_2 x$$

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$$X'(x) = c_2$$

$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(1,t) = 0$ imply that $X'(0) = 0$ and $X'(1) = 0$ so $c_2 = 0$

$$X(x) = c_1 = \text{Constant}$$

$$X''(x) + \lambda^2 X(x) = 0$$

If $\lambda \neq 0$ then:

$$X(x) = A \sin \lambda x + B \cos \lambda x$$

$$X(x) = A\sin\lambda x + B\cos\lambda x$$

$$X'(x) = A\lambda\cos\lambda x - B\lambda\sin\lambda x$$

$$X'(0) = 0 = A\lambda\cos 0 - B\lambda\sin 0 = A\lambda$$

$$0 = A$$

If $A = 0$ then:

$$X(x) = A \sin \lambda x + B \cos \lambda x = B \cos \lambda x$$

$$X'(x) = -B\lambda \sin \lambda x$$

$$X'(1) = 0 = -B\lambda \sin \lambda$$

$$0 = \sin \lambda$$

$$\lambda = \pi,\ 2\pi,\ 3\pi, \cdots$$

$$\lambda = \pi n \text{ where } n = 0,\ 1,\ 2,\ 3,\ \cdots$$

$$T(t) = (\text{const})e^{-\alpha^2\lambda^2 t} = (\text{const})e^{-\alpha^2\pi^2 n^2 t}$$

If $n \geq 1$ then $X(x) = (\text{const}) \cos n\pi x$

If $n = 0$ then $X(x) = \text{const}$

$$\lambda = \pi n \text{ where } n = 0,\ 1,\ 2,\ 3,\ \cdots$$

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$X(x) = (\text{const})\cos n\pi x$ for $n = 0,\ 1,\ 2,\ \cdots$

$$X(x)T(t) = (\text{const})e^{-(\alpha\pi n)^2 t}\cos n\pi x$$

Any expression of this type will solve the partial differential equation and the boundary conditions.

General solution:

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(\alpha\pi n)^2 t} \cos n\pi x$$

The initial condition $u(x,0) = \sin^2 \pi x$ determines the values of a_n

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(\alpha\pi n)^2 t} \cos n\pi x$$

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos n\pi x$$

$$\sin^2 \pi x = \sum_{n=0}^{\infty} a_n \cos n\pi x$$

$$a_n = \int_{-1}^{1} \sin^2 \pi x \cos n\pi x \; dx$$

$$\sin^2 \pi x = \sum_{n=0}^{\infty} a_n \cos n\pi x$$

$$\frac{1}{2}(1 - \cos 2\pi x) = \sum_{n=0}^{\infty} a_n \cos n\pi x$$

$$\frac{1}{2} - \frac{1}{2} \cos 2\pi x = \sum_{n=0}^{\infty} a_n \cos n\pi x$$

We want

$$\frac{1}{2} - \frac{1}{2}\cos 2\pi x$$

to be the same as:

$$a_0 + a_1 \cos 1\pi x + a_2 \cos 2\pi x + a_3 \cos 3\pi x + \cdots$$

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$a_0 = \frac{1}{2}$, $a_1 = 0$, $a_2 = -\frac{1}{2}$, $a_3 = 0$

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \cos n\pi x \\
&= \frac{1}{2} - \frac{1}{2} e^{-4\alpha^2\pi^2 t} \cos 2\pi x
\end{aligned}
$$

$$u(x,t) = \frac{1}{2} - \frac{1}{2}e^{-4\alpha^2\pi^2 t}\cos 2\pi x$$