Laplace's Equation

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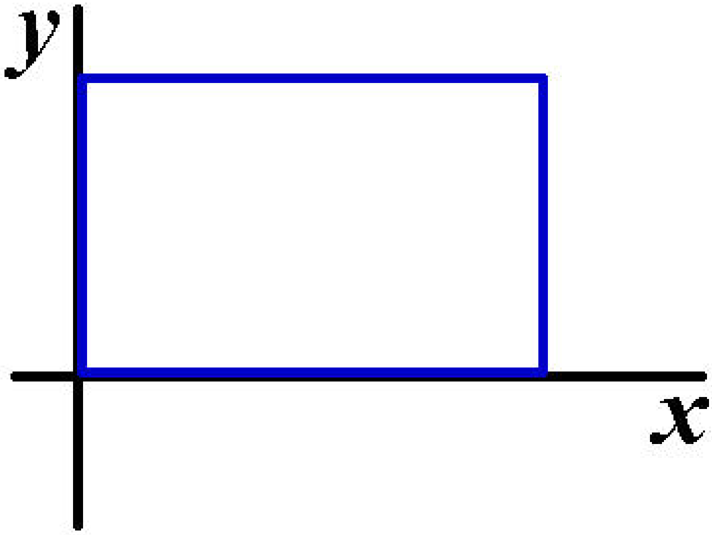
$$u = u(x,t)$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

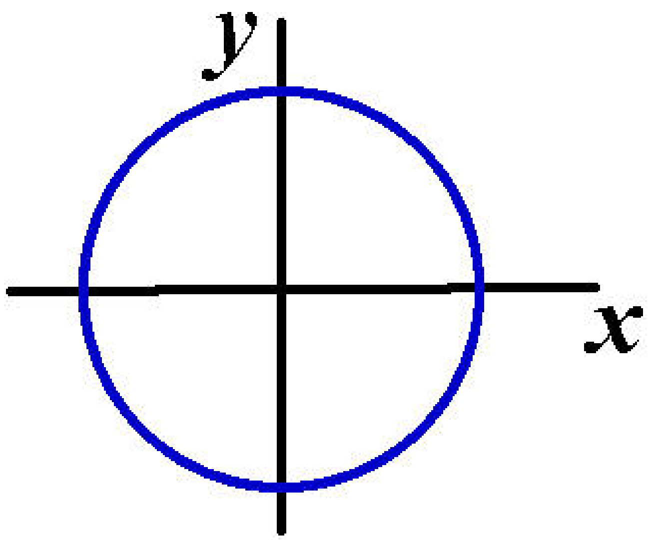
$$u = u(x, y, t)$$

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$



$$u = u(r, \theta, t)$$

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} \right)$$



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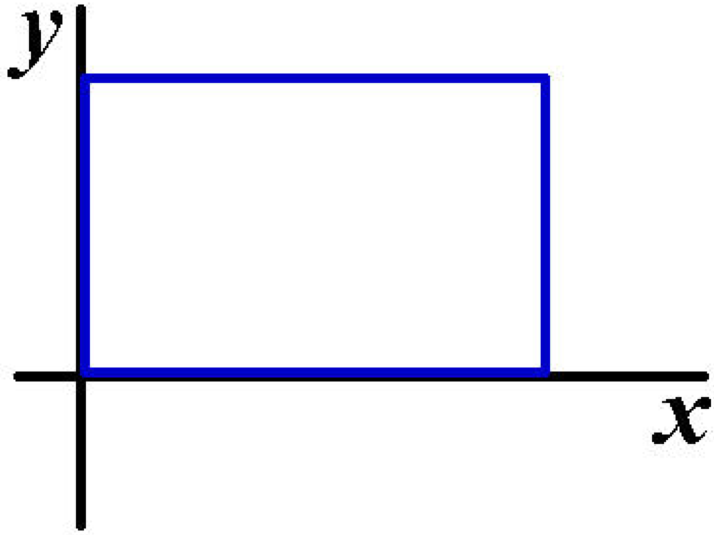
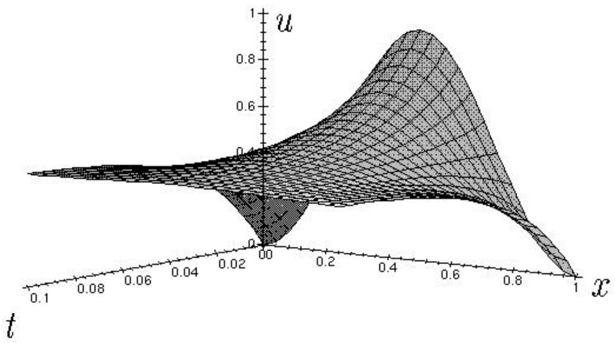
$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$



At Steady-State: $$\frac{\partial u}{\partial t} = 0$$



$$u = u(x, y, t)$$

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

At steady-state $\frac{\partial u}{\partial t} = 0$ so:

$$0 = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

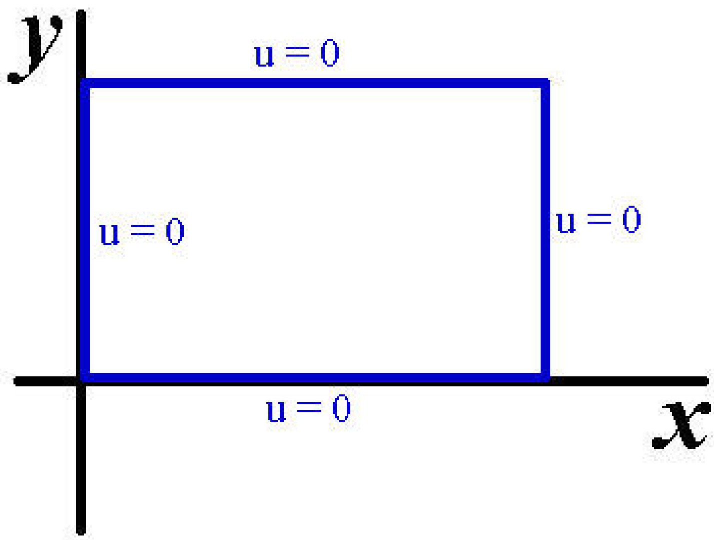
$$u = u(x, y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Laplace's Equation

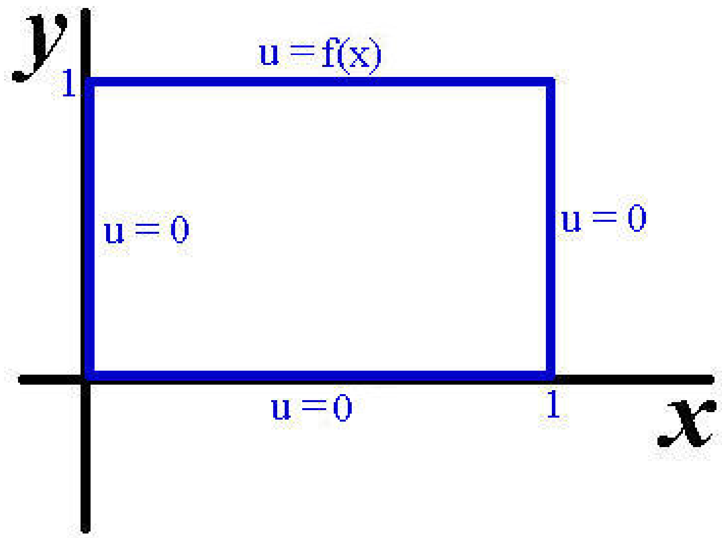
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If u is 0 on all boundaries, then the temperature in the interior is 0 at all points. This is the trivial solution.



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

For a nontrivial solution, we need u to be nonzero on at least one boundary.

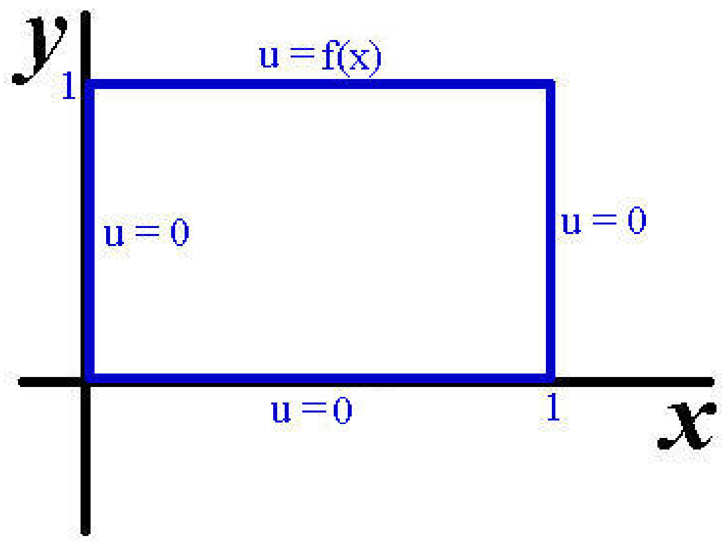


$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Boundary conditions:

$$u(x,1) = f(x) \qquad u(0,y) = 0$$

$$u(x,0) = 0 \qquad u(1,y) = 0$$



Find all functions $X(x)$ and $Y(y)$ so that $X(x)Y(y)$ solves:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Find all functions $X(x)$ and $Y(y)$ so that $X(x)Y(y)$ solves:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)}$$

Find all functions $X(x)$ and $Y(y)$ so that $X(x)Y(y)$ solves:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = \lambda$$

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = \lambda$$

$$-\frac{X''(x)}{X(x)} = \lambda \qquad \frac{Y''(y)}{Y(y)} = \lambda$$

$$X''(x) + \lambda X(x) = 0 \qquad Y''(y) - \lambda Y(y) = 0$$

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = 0 \qquad X(1) = 0$$

If $\lambda = 0$, we will get only the trivial solution

$$X(x) = 0 \text{ for all } x$$

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = 0 \qquad X(1) = 0$$

If $\lambda \neq 0$ then:

$$X(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$$

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = 0 \qquad X(1) = 0$$

If $\lambda \neq 0$ then:

$$X(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$$

$$X(0) = 0 = A \cos 0 + B \sin 0 = A$$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

If $A = 0$ then:

$$X(x) = B\sin\sqrt{\lambda}x$$

$$X(1) = 0 = B\sin\sqrt{\lambda}$$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

If $A = 0$ then:

$$X(x) = B\sin\sqrt{\lambda}x$$

$$X(1) = 0 = B\sin\sqrt{\lambda}$$

$$0 = \sin\sqrt{\lambda}$$

$$\sqrt{\lambda} = n\pi \text{ where } n = 1,\ 2,\ 3,\ \cdots$$

If $\sqrt{\lambda} = n\pi$ then:

$$X(x) = B \sin \sqrt{\lambda} x = B \sin n\pi x$$

Now, solve:

$$Y''(y) - \lambda Y(y) = 0$$

$$Y''(y) - n^2\pi^2 Y(y) = 0$$

$$Y''(y) - n^2\pi^2 Y(y) = 0$$

Substitute e^{ry}

$$r^2 e^{ry} - n^2\pi^2 e^{ry} = 0$$

$$r^2 - n^2\pi^2 = 0$$

$$r = \pm n\pi$$

$$Y''(y) - n^2\pi^2 Y(y) = 0$$

e^{ry} is a solution if $r = n\pi$ or $r = -n\pi$

$e^{n\pi y}$ and $e^{-n\pi y}$ are solutions

$$Y(y) = c_1 e^{n\pi y} + c_2 e^{-n\pi y}$$

$$Y(y) = c_1 e^{n\pi y} + c_2 e^{-n\pi y}$$

The boundary condition $Y(0) = 0$ implies:

$$0 = c_1 + c_2$$

$$c_2 = -c_1$$

$$Y(y) = c_1 e^{n\pi y} - c_1 e^{-n\pi y}$$

$$\sinh\theta = \frac{1}{2}\left(e^{\theta} - e^{-\theta}\right)$$

$$2\sinh\theta = e^{\theta} - e^{-\theta}$$

$$\begin{aligned} Y(y) &= c_1 e^{n\pi y} - c_1 e^{-n\pi y} \\ &= c_1\left(e^{n\pi y} - e^{-n\pi y}\right) \\ &= 2c_1 \sinh n\pi y \\ &= (\text{const}) \sinh n\pi y \end{aligned}$$

$$X(x) = (\text{const}) \sin n\pi x$$

$$Y(y) = (\text{const}) \sinh n\pi y$$

Therefore:

$$X(x)Y(y) = (\text{const}) \sin n\pi x \sinh n\pi y$$

What about the last remaining boundary condition $u(x, 1) = f(x)$

$$X(x)Y(y) = (\text{const}) \sin n\pi x \sinh n\pi y$$

$$u(x, 1) = f(x)$$

$$f(x) = X(x)Y(1) = (\text{const}) \sin n\pi x \sinh n\pi$$

Can't solve!

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh n\pi y \sin n\pi x$$

Now impose the last boundary condition: $u(x, 1) = f(x)$

$$f(x) = u(x, 1) = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin n\pi x$$

$$f(x) = u(x,1) = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin n\pi x$$

$$\begin{aligned} b_n \sinh n\pi &= \int_{-1}^{1} f(x) \sin n\pi x \, dx \\ &= 2 \int_{0}^{1} f(x) \sin n\pi x \, dx \end{aligned}$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh n\pi y \sin n\pi x$$

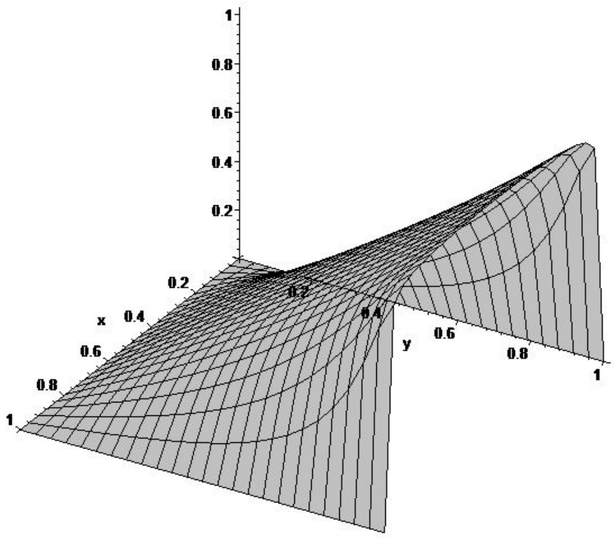
$$b_n = \frac{2}{\sinh n\pi} \int_0^1 f(x) \sin n\pi x \, dx$$

Example: Suppose $f(x) = 1$

$$\begin{aligned} b_n &= \frac{2}{\sinh n\pi} \int_0^1 \sin n\pi x \, dx \\ &= \frac{2}{\sinh n\pi} \left[\frac{-1}{n\pi} \cos n\pi x \right]_0^1 \\ &= \frac{2}{n\pi \sinh n\pi} \left(1 - (-1)^n\right) \end{aligned}$$

$u(x, y)$ is given by:

$$\sum_{n=1}^{\infty} \frac{2\left(1-(-1)^{n}\right)}{n\pi \sinh n\pi} \sinh n\pi y \sin n\pi x$$



1
0.8
0.6
0.4
0.2
x
0.2
0.4
0.6
0.8
1
y
0.2
0.4
0.6
0.8
1