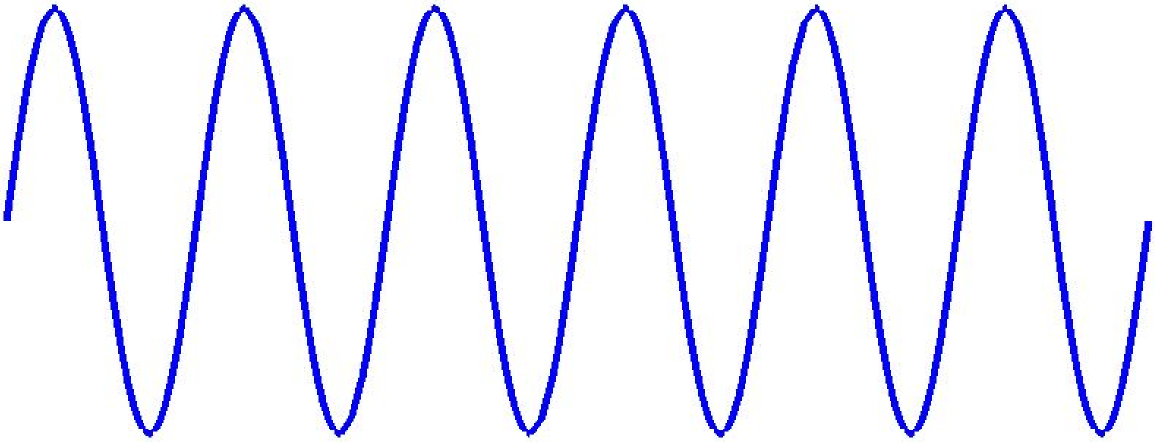
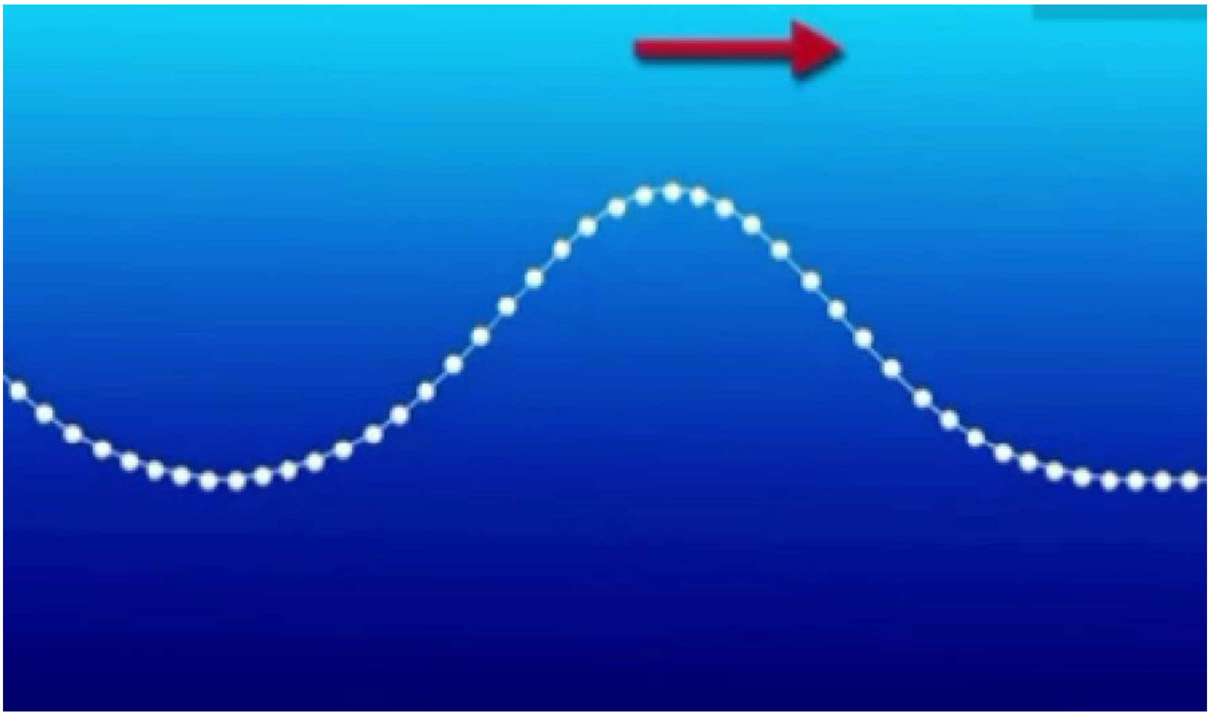
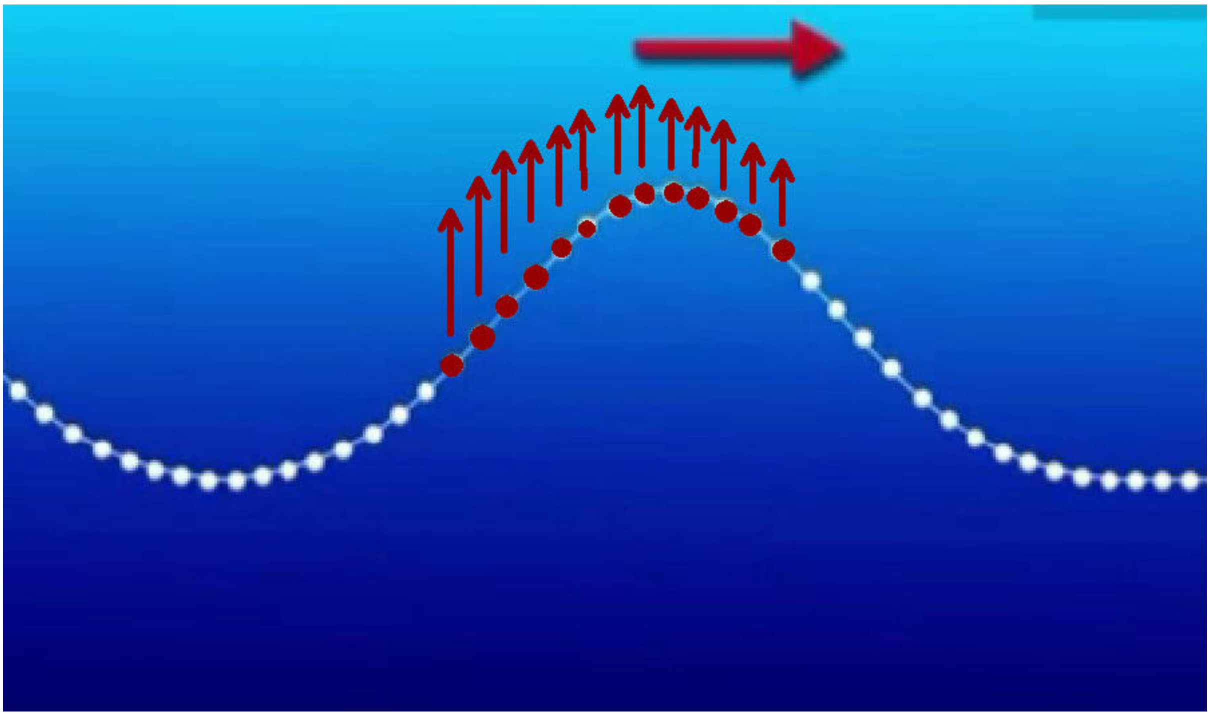
The Wave Equation

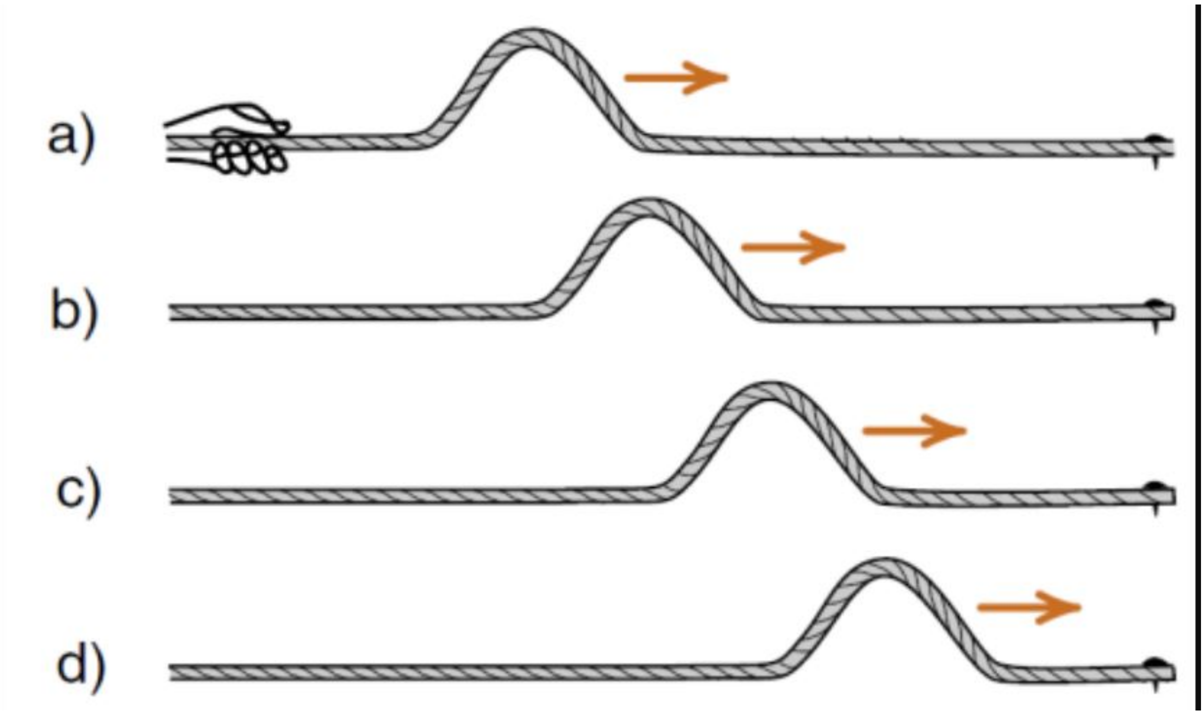
Dr. Elliott Jacobs

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

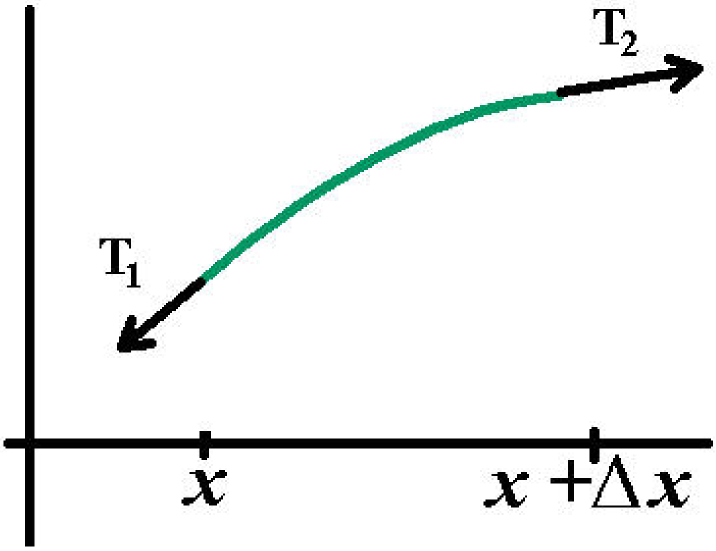




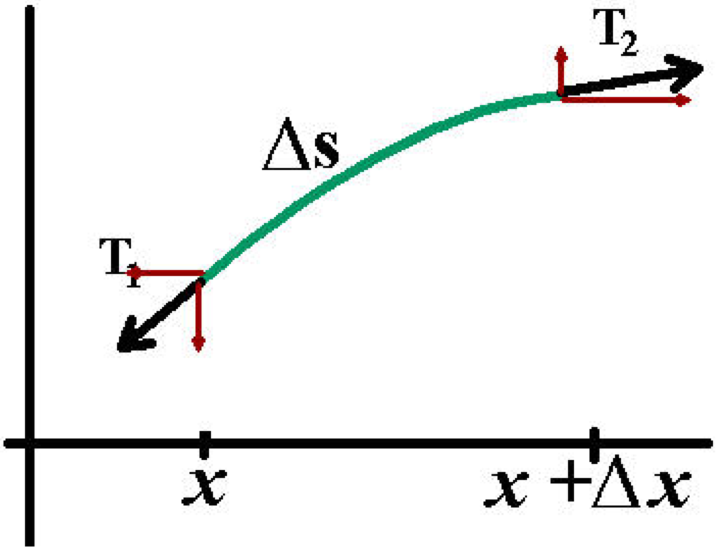



a)
b)
c)
d)

$$u = u(x,t)$$



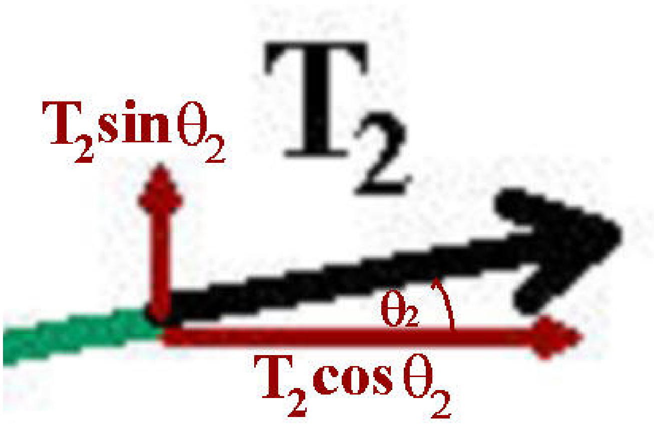
$$u = u(x,t)$$



At the right end:

$$\text{Horizontal Component} = T_2 \cos\theta_2$$

$$\text{Vertical Component} = T_2 \sin\theta_2$$



At the right end:

$$\text{Horizontal Component} = T_2 \cos\theta_2$$

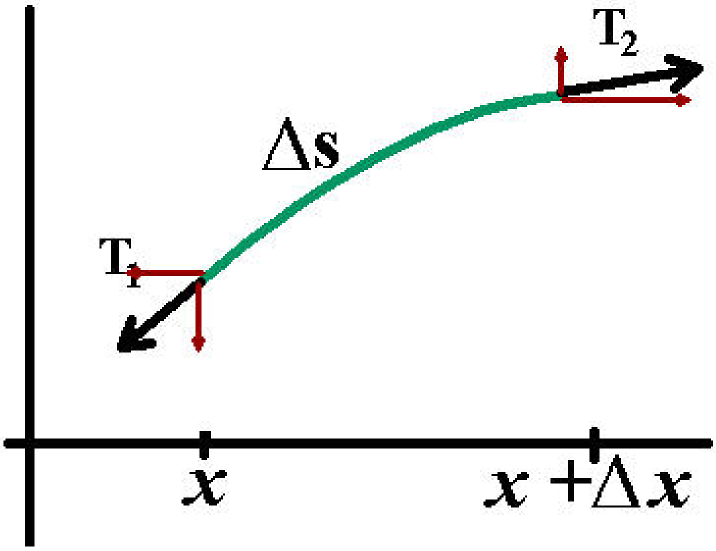
$$\text{Vertical Component} = T_2 \sin\theta_2$$

At the left end:

$$\text{Horizontal Component} = T_1 \cos\theta_1$$

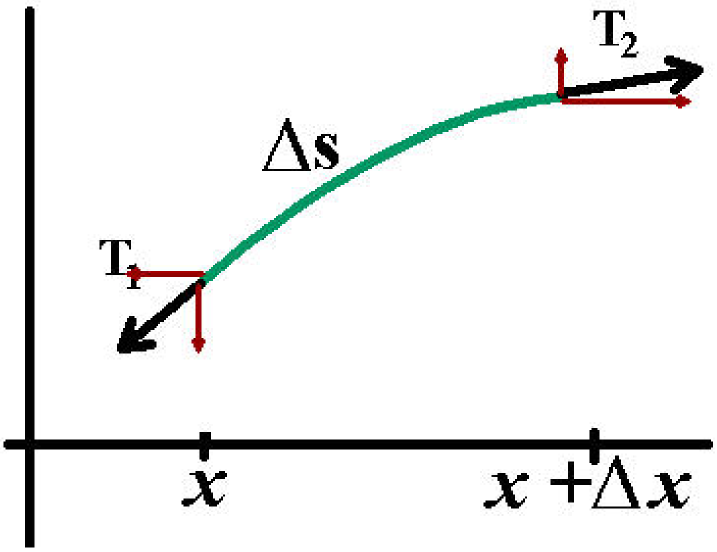
$$\text{Vertical Component} = T_1 \sin\theta_1$$

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T$$



$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = (\text{mass})(\text{acceleration})$$

$$\text{mass} = \rho \Delta s \qquad \text{acceleration} = \frac{\partial^2 u}{\partial t^2}$$



$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = \rho \Delta s \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\theta_2}{T} - \frac{T_1 \sin\theta_1}{T} = \frac{\rho}{T} \Delta s \frac{\partial^2 u}{\partial t^2}$$

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T$$

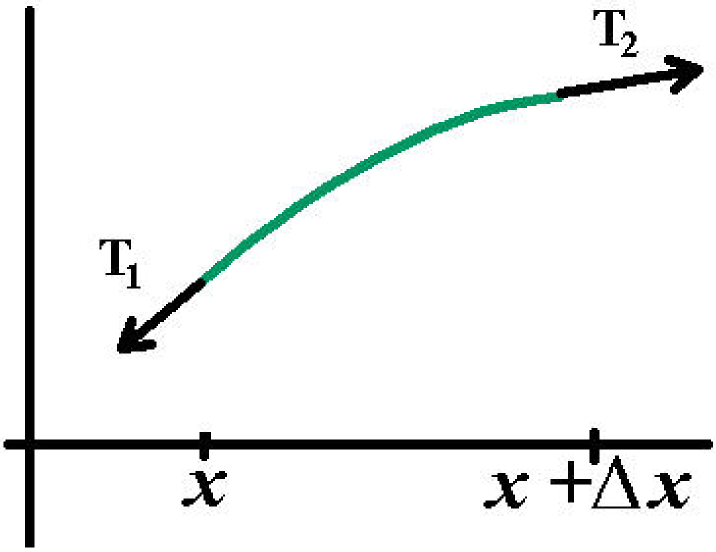
$$\frac{T_2 \sin\theta_2}{T} - \frac{T_1 \sin\theta_1}{T} = \frac{\rho}{T}\Delta s \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} = \frac{\rho}{T}\Delta s \frac{\partial^2 u}{\partial t^2}$$

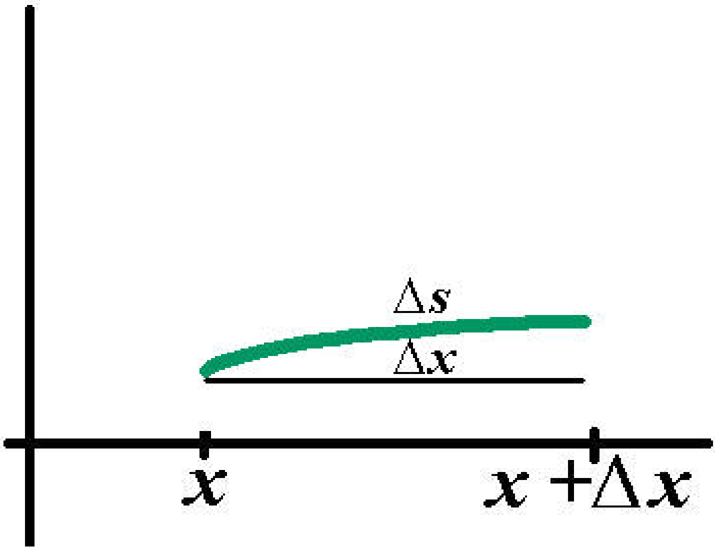
$$\tan\theta_2 - \tan\theta_1 = \frac{\rho}{T}\Delta s \frac{\partial^2 u}{\partial t^2}$$

$$\tan\theta_2 - \tan\theta_1 = \frac{\rho}{T}\Delta s\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial u}{\partial x}(x+\Delta x,\ t) - \frac{\partial u}{\partial x}(x,\ t) = \frac{\rho}{T}\Delta s\frac{\partial^2 u}{\partial t^2}$$



For a small deflection: $\Delta s \approx \Delta x$



$$\frac{\partial u}{\partial x}(x+\Delta x,\ t)-\frac{\partial u}{\partial x}(x,\ t)\approx\frac{\rho}{T}\Delta x\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\frac{\partial u}{\partial x}(x+\Delta x,\ t)-\frac{\partial u}{\partial x}(x,\ t)}{\Delta x}\approx\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\frac{\partial u}{\partial x}(x + \Delta x,\ t) - \frac{\partial u}{\partial x}(x,\ t)}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Take the limit as Δx goes to 0

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{T}{\rho} \qquad : \qquad \frac{\text{kg m/sec}^2}{\text{kg/m}} = \frac{\text{m}^2}{\text{sec}^2}$$

$$\text{Let } c = \sqrt{\frac{T}{\rho}}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

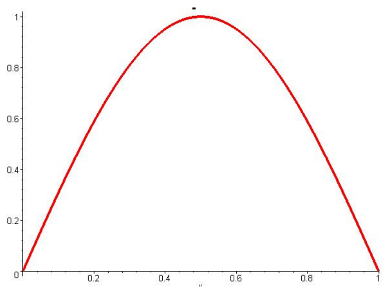
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions:

$$u(0,t) = u(1,t) = 0$$

Initial conditions:

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = 0$$



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions:

$u(0, t) = u(1, t) = 0$

Initial conditions:

$u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = 0$

Substitute $T(t)X(x)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Substitute $T(t)X(x)$

$$T''(t)X(x) = c^2 T(t)X''(x)$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Substitute $T(t)X(x)$

$$T''(t)X(x) = c^2 T(t)X''(x)$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

$$\frac{T''(t)}{c^2T(t)} = -\lambda^2 \qquad \frac{X''(x)}{X(x)} = -\lambda^2$$

$$T''(t) + c^2\lambda^2 T(t) = 0$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

$$X(0) = 0 \quad \text{implies} \quad A = 0$$

$$X(x) = B\sin\lambda x$$

$$X(1) = 0 \quad \text{implies} \quad 0 = B\sin\lambda$$

$$0 = \sin\lambda$$

$$\lambda = n\pi \quad \text{where } n = 1,\ 2,\ 3,\ \cdots$$

$$X(x) = B \sin n\pi x$$

$$T''(t) + c^2\lambda^2 T(t) = 0$$

$$T''(t) + c^2 n^2 \pi^2 T(t) = 0$$

$$T(t) = \alpha \cos n\pi ct + \beta \sin n\pi ct$$

Initial condition: $u(x, 0) = f(x)$

$$X(x)T(0) = f(x) \quad \text{(may be inconsistent)}$$

$$T''(t) + c^2 n^2 \pi^2 T(t) = 0$$

$$T(t) = \alpha \cos n\pi ct + \beta \sin n\pi ct$$

Other initial condition: $\frac{\partial u}{\partial t}(x, 0) = 0$.
If $u = X(x)T(t)$ then:

$$X(x)T'(0) = 0$$

$$T'(0) = 0$$

$$T'(0) = 0$$

$$T(t) = \alpha \cos n\pi ct + \beta c \sin n\pi ct$$

$$T'(t) = -c\alpha n\pi \sin n\pi ct + c\beta n\pi \cos n\pi ct$$

$$T'(0) = c\beta n\pi = 0$$

$$\beta = 0$$

$$T(t) = \alpha \cos n\pi ct$$

$$X(x) = (\text{const}) \sin n\pi x$$

$$T(t) = (\text{const}) \cos n\pi ct$$

$$X(x)T(t) = (\text{const}) \cos(n\pi ct) \sin(n\pi x)$$

where $n = 1, 2, 3, \ldots$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(n\pi ct) \sin(n\pi x)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(n\pi c t)\sin(n\pi x)$$

Initial amplitude $u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(n\pi ct) \sin(n\pi x)$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \cos(0) \sin(n\pi x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx$$

Let's take a very simple initial function:

$$f(x) = \sin \pi x$$

Let's also take $c = 1$

$$\sin \pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$= b_1 \sin \pi x + b_2 \sin 2\pi x + b_3 \sin 3\pi x + \cdots$$

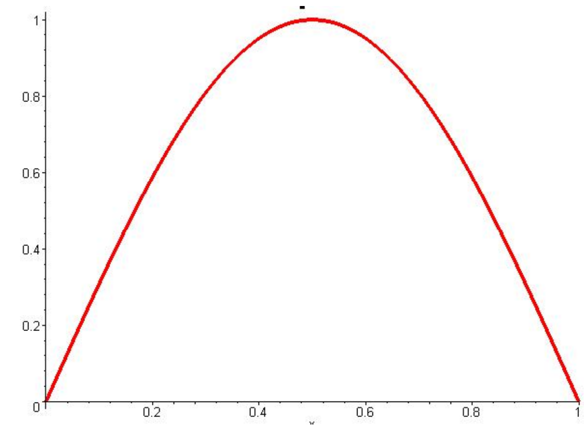
$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(n\pi t) \sin(n\pi x)$$

If $b_1 = 1$ and $b_n = 0$ for $n > 1$ then:

$$u(x,t) = \cos(\pi t) \sin(\pi x)$$

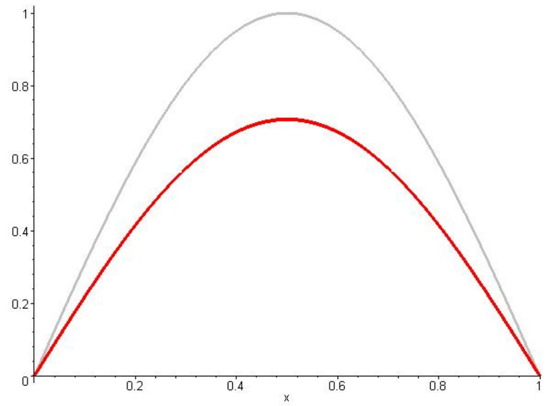
$$u(x,t) = \cos(\pi t)\sin(\pi x)$$

Graph of $u(x,0)$:



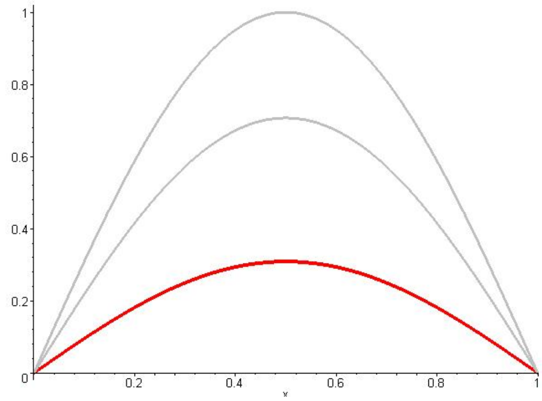
$$u(x,t) = \cos(\pi t)\sin(\pi x)$$

Graph of $u\left(x, \frac{1}{4}\right)$:



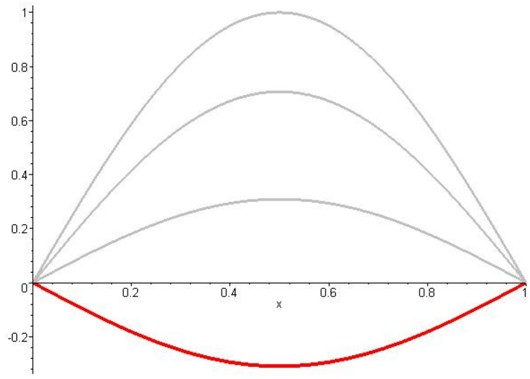
$$u(x,t) = \cos(\pi t)\sin(\pi x)$$

Graph of $u\left(x, \frac{1}{2}\right)$:



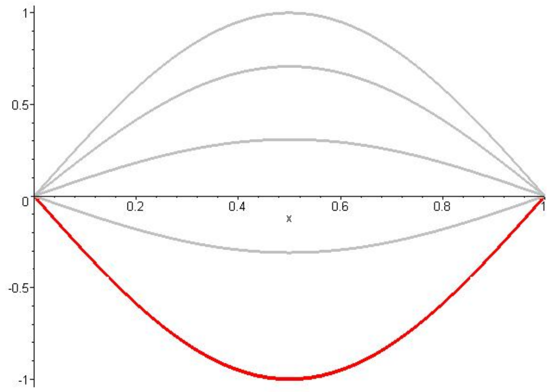
$$u(x,t) = \cos(\pi t)\sin(\pi x)$$

Graph of $u\left(x, \frac{3}{4}\right)$:



$$u(x,t) = \cos(\pi t)\sin(\pi x)$$

Graph of $u(x,1)$:



$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(nc\pi t) \sin(n\pi x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

Add:

$$\sin(\theta + \phi) + \sin(\theta - \phi) = 2\sin\theta\cos\phi$$

$$\sin\theta\cos\phi = \frac{1}{2}(\sin(\theta+\phi)+\sin(\theta-\phi))$$

$$\sin n\pi x\cos n\pi ct = \frac{1}{2}(\sin(n\pi(x+ct))+\sin(n\pi(x-ct)))$$

$$\sin n\pi x \cos n\pi ct = \frac{1}{2}(\sin(n\pi(x+ct)) + \sin(n\pi(x-ct)))$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(n\pi ct) \sin(n\pi x)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty} b_n \sin(n\pi(x+ct)) + \sum_{n=1}^{\infty} b_n \sin(n\pi(x-ct))\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$f(x+ct) = \sum_{n=1}^{\infty} b_n \sin(n\pi(x+ct))$$

$$f(x-ct) = \sum_{n=1}^{\infty} b_n \sin(n\pi(x-ct))$$

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n \cos(n\pi ct)\sin(n\pi x) \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty} b_n \sin(n\pi(x+ct)) + \sum_{n=1}^{\infty} b_n \sin(n\pi(x-ct))\right) \\
&= \frac{1}{2}(f(x+ct) + f(x-ct))
\end{aligned}
$$

d'Alembert's Formula

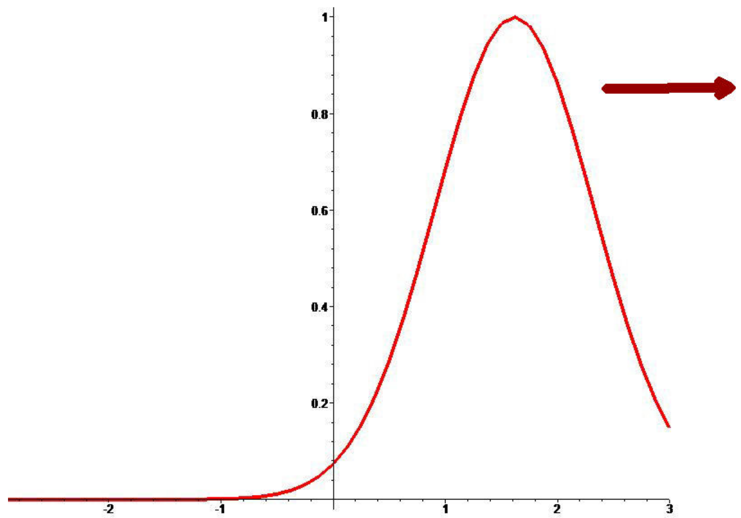
$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct))$$

Example:

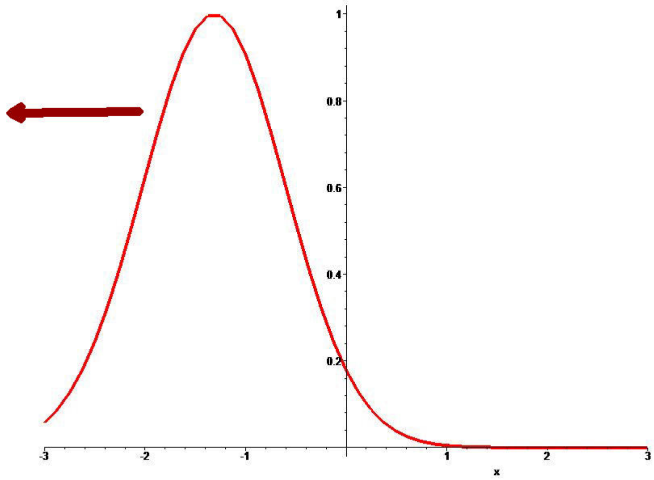
Take $f(x) = e^{-x^2}$

What is the meaning of $f(x-ct)$?

$f(x - ct)$ represents motion of a wave to the right at speed c



$f(x + ct)$ represents motion of a wave to the right at speed c



d'Alembert's Formula

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct))$$